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On some consequences of the canonical transformation in the Hamiltonian theory of water waves

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We discuss some consequences of the canonical transformation in the Hamiltonian theory of water waves (Zakharov, *J. Appl. Mech. Tech. Phys.*, vol. 9, 1968, pp. 190–194). Using Krasitskii's canonical transformation we derive general expressions for the second-order wavenumber and frequency spectrum and the skewness and the kurtosis of the sea surface. For deep-water waves, the second-order wavenumber spectrum and the skewness play an important role in understanding the so-called sea-state bias as seen by a radar altimeter. According to the present approach but in contrast with results obtained by Barrick & Weber (*J. Phys. Oceanogr.*, vol. 7, 1977, pp. 11–21), in deep water second-order effects on the wavenumber spectrum are relatively small. However, in shallow water in which waves are more nonlinear, the second-order effects are relatively large and help to explain the formation of the observed second harmonics and infra-gravity waves in the coastal zone. The second-order effects on the directional-frequency spectrum are as a rule more important; in particular it is shown how the Stokes-frequency correction affects the shape of the frequency spectrum, and it is also discussed why in the context of the second-order theory the mean-square slope cannot be estimated from time series. The kurtosis of the wave field is a relevant parameter in the detection of extreme sea states. Here, it is argued that in contrast perhaps to one's intuition, the kurtosis decreases while the waves approach the coast. This is related to the generation of the wave-induced current and the associated change in mean sea level.

1. Introduction

Surface gravity waves are usually described in the context of the potential flow of an ideal fluid. As discovered by Zakharov (1968), the resulting nonlinear evolution equations can be obtained from a Hamiltonian, which is the total energy of the fluid, while the appropriate canonical variables are the surface elevation $\eta(\boldsymbol{x}, t)$ and the value ψ of the potential ϕ at the surface, $\psi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, z = \eta, t))$.

For small wave steepness the potential inside the fluid may be expressed in an approximate manner in terms of the canonical variables, and as a result the Hamiltonian becomes a series expansion in terms of the action variable $A(\boldsymbol{k}, t)$ (which is related to the Fourier transform of the canonical variables). The second-order term corresponds then to linear theory, while the third- and fourth-order terms represent effects of three- and four-wave interactions. Excluding effects of capillarity,

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it is well known that the dispersion relation for surface gravity waves does not allow resonant three-wave interactions, and as a consequence there exists a non-singular canonical transformation of the type

$$A = A(a, a^*)$$

that allows the elimination of the third-order terms from the Hamiltonian. In terms of the new action variable $a(\boldsymbol{k}, t)$ the Hamiltonian now only has quadratic and quartic terms, and the Hamilton equation attains a relatively simple form and is known as the Zakharov equation.

The properties of the Zakharov equation have been studied in great detail by, for example, Crawford *et al.* (1981), Yuen & Lake (1982) and Krasitskii & Kalmykov (1993). Thus the nonlinear dispersion relation, first obtained by Stokes (1947), follows from the Zakharov equation and also the instability of a weakly nonlinear, uniform wavetrain (the so-called Benjamin–Feir instability); the results on growth rates, for example, are in good agreement with the results by Longuet-Higgins (1978), who did a numerical study of the instabilities of deep-water waves in the context of the exact equations. It is noted that once the solution to the Zakharov equation is known for a, one still needs to apply the canonical transformation to recover the actual action variable A and hence the surface elevation. Although the difference between the two action variables is only of the order of the wave steepness, explaining why relatively less attention has been devoted to the consequences of the canonical transformation, there are a number of applications where one is interested in the effects of bound waves. Examples are the high-frequency radar (e.g. Wyatt 2000), which basically measures aspects of the second-order spectrum, and the estimation of the sea-state bias as seen by an altimeter (Elfouhaily *et al.* (1999).

In this paper I would like to study some properties and consequences of the canonical transformation in the context of the statistical theory of weakly nonlinear ocean waves. For small wave steepness one finds from the Zakharov equation that at the lowest order the action variable $a(\boldsymbol{k}, t)$ follows a linear evolution equation; hence the action variable obeys Gaussian statistics. The (nearly) Gaussian property of the ocean surface follows from the central limit theorem which tells us that if the waves have random and independent phase, then the probability distribution is Gaussian. The waves are to a good approximation independent because they have propagated into a given area of the ocean from different distant regions. Even if initially one would start with a highly correlated state, then, because of dispersion, wave groups separate, thereby decreasing the correlation. On the other hand, finite-steepness waves may give rise to correlations between the different wave components because of (resonant) wave–wave interactions. However, the effect is small for small steepness. Therefore, in practice one nearly always finds that for dispersive ocean waves the Gaussian property holds in good approximation. For a more detailed discussion see Hasselmann (1967).

Then, given the Gaussian property of the sea surface, effects of nonlinearity on the moments of the surface elevation may be evaluated using the canonical transformation. As a first example, I consider the second moment $\langle \eta^2 \rangle$ and the associated wavenumber-variance spectrum $F(\boldsymbol{k})$ and directional-frequency spectrum $F(\Omega, \theta)$. The second-order corrections to the wave spectrum (called the second-order spectrum for short) are obtained by deriving a general expression for the wavenumber–frequency spectrum. The wavenumber spectrum and the frequency spectrum then follow from the marginal-distribution laws. Some of the properties of these second-order spectra are discussed in some detail, both for deep water and for shallow water.

Regarding the wavenumber spectrum it is shown that the second-order spectrum is small compared to the first-order spectrum. This contrasts with Barrick & Weber (1977) whose work indicates that for large wavenumbers the perturbation expansion diverges. However, following Creamer *et al.* (1989) it is argued here that Barrick & Weber (1977) overlooked an important, quasi-linear term which removes the divergent behaviour of the second-order spectrum. Creamer *et al.* (1989) considered improved representations of ocean surface waves using a Lie transformation and applied their work to the determination of the second-order spectrum in one dimension. Our results on the second-order spectrum, although obtained via the different route of Krasitskii's canonical transformation, are in complete agreement with Creamer *et al.* (1989), but our explicit result is slightly more general, as it holds for two-dimensional propagation and also in waters of finite depth. It is worthwhile to mention that Zakharov (1992) and Krasitskii (1994) considered the slightly simpler problem of the higher-order corrections to the action-density spectrum. They found that the second-order action-density spectrum contains two groups of terms, namely terms which are fully nonlinear and terms which are termed quasi-linear because they are proportional to the first-order action spectrum. The quasi-linear terms are an example of self-interaction and give a nonlinear correction to the action or energy of the free waves, whereas the fully nonlinear terms describes the amount of energy of the bound waves which do not satisfy the linear dispersion relation.

While the second-order wavenumber spectrum consists of two contributions, namely one contribution giving the effects of bound waves and one the quasi-linear term, the second-order frequency spectrum has an additional term which, not surprisingly, is related to the Stokes-frequency correction. In deep water the Stokes-frequency correction is positive and therefore gives an upshift of the peak of the frequency spectrum. However, this upshift is small compared to the downshift of the spectrum caused by the resonant four-wave interactions. But the second-order corrections do have an impact on the high-frequency tail of the spectrum. Taking as the first-order spectrum a Phillips spectrum which has an Ω^{-5} tail, it is found that from twice the peak frequency onward the sum of the first- and second-order spectra (hereafter called the total spectrum) has approximately an Ω^{-4} shape. Hence, the second-order corrections to the frequency spectrum are important, and they mainly stem from the combined effects of the generation of bound waves and the quasi-linear self-interaction.

In shallow water, gravity waves are typically more nonlinear, as the ratio of the amplitude of the second harmonic to the first harmonic rapidly increases with decreasing dimensionless depth. Therefore, compared to the first-order spectrum the second-order spectrum may give rise to considerable contributions, in particular in the frequency domain around twice the peak frequency and in the low-frequency range in which forced infra-gravity waves are generated. In addition, for a dimensionless depth $O(1)$, the Stokes-frequency correction is found to give a downshift of the peak of the frequency spectrum. This downshift is considerable, also compared to the shift in the spectral peak caused by the resonant four-wave interactions (Janssen & Onorato, 2007).

As a second example I consider the determination of the skewness and the kurtosis of the sea surface. The skewness parameter is important when one is interested in the determination of the sea-state bias as experienced by a radar altimeter onboard a satellite (see e.g. Srokosz 1986), while the kurtosis is an important parameter to assess whether there is an increased probability of an extreme sea state, e.g. the likely occurrence of freak waves (Janssen 2003). In particular, the dependence of these statistical parameters on the spectral shape and the dimensionless depth is studied.

Regarding the depth dependence, the important role of the wave-induced mean sea level is pointed out. In the presence of wave groups finite-amplitude ocean waves give rise to a set-down, and as a consequence the skewness and kurtosis parameters are reduced to a considerable extent. This has important consequences for the occurrence of extreme events in shallow water.

The programme of this paper is as follows. After giving some background on the reason why this study was started, § 2 gives a brief overview of the Hamiltonian theory of surface gravity waves, while in § A 1 a detailed derivation of the canonical transformation is presented. In § 3 the general expression of the wavenumber–frequency spectrum is obtained in terms of the coefficients of the canonical transformation. The wavenumber and the directional-frequency spectrum then follow immediately from the marginal distribution laws. Section 3 shows that the total-wavenumber spectrum agrees with the deep-water result of Creamer *et al.* (1989), highlighting the important role of the quasi-linear term. Also, some interesting properties of the second-order frequency spectrum for both deep water and water of finite depth are discussed. In particular, the deep-water frequency spectra have a fatter tail due to the bound waves, which gives rise to a considerable overestimatation of the mean-square slope. Furthermore, in shallow water the Stokes-frequency correction results in a sizeable downshift of the peak of the spectrum. In § 4 the skewness and the kurtosis are determined for general spectra, and the dependence of the statistical parameters on the depth and the spectral shape is briefly studied. Conclusions are presented in § 5.

As the development presented here is fairly elaborate, § A 3 gives all the relevant results starting from the canonical transformation of a single wavetrain, and these single-mode results have been used as a check on the general results of the main text. A preliminary account of this work may be found in Janssen (2004).

1.1. *Background*

This investigation started when it was realized that according to the work of Barrick & Weber (1977) the weakly nonlinear perturbation expansion for surface gravity waves is not convergent. For small wave steepness the nonlinear evolution equations have been solved by means of a perturbation expansion by several authors (Tick 1959; Longuet-Higgins 1963; Barrick & Weber 1977), which allows to write down an expression for the second-order correction to the wavenumber–frequency spectrum $F(k, \omega)$. By integrating $F(k, \omega)$ over angular frequency, the following elegant result for the total-wavenumber spectrum $F(k)$ is found:

$$F(k) = E(k) + \frac{1}{2}k^2 \int_{k/2}^{\infty} \mathrm{d}k' \; E(k')E(|k-k'|), \tag{1}$$

where $E(k)$ is the first-order spectrum.

It is instructive to determine $F(k)$ for a simple input spectrum $E(k)$. For the Phillips spectrum

$$E(k) = \frac{1}{2}\alpha_p k^{-3}, \;\; k \geqslant k_0, \tag{2}$$

with k_0 the peak wavenumber and α_p the Phillips parameter, the result is

$$F(k) = E(k) + \frac{1}{8}\alpha_p^2 \left[\frac{6}{k^3} \log\left(\frac{k^2}{k_0^2} - 1\right) + \frac{k^2}{k_0^2} \left\{ \frac{6}{k^3} - \frac{1}{k}\frac{k^2+k_0^2}{\left(k^2-k_0^2\right)^2} - \frac{4}{k\left(k^2-k_0^2\right)} \right\} \right] \tag{3}$$

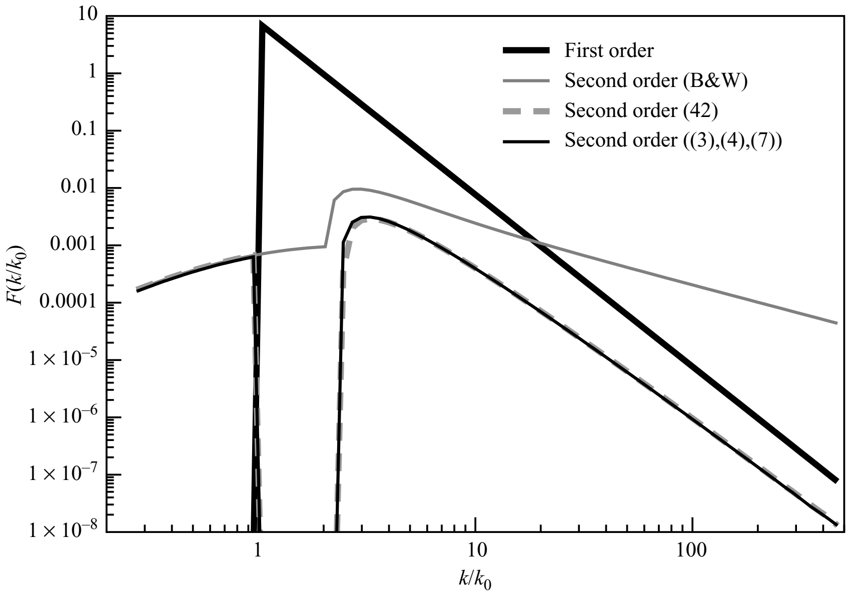

FIGURE 1. The second-order effects on the surface-wave-height spectrum, illustrating the importance of the quasi-linear term.

for $k > 2k_0$, while for $k < 2k_0$ one has

$$F(k) = E(k) + \frac{1}{8}\alpha_p^2\left[\frac{6}{k^3}\log\left(\frac{k}{k_0}+1\right) + \frac{k^2}{k_0^2}\left\{\frac{3}{k^3} - \frac{6k_0}{k^4} - \frac{\frac{5}{2}k + \frac{1}{2}k_0}{k^2(k+k_0)^2}\right\}\right]. \tag{4}$$

A plot of this special case is given in figure 1, and the present result is labelled B&W. It is striking that for large k the second-order spectrum dominates the first-order spectrum. This is undesirable because it signals that the perturbation approach may not be convergent. As a consequence, parameters such as the wave variance m_0 and the mean-square slope mss defined by

$$m_0 = \int \mathrm{d}k\ F,\ \ mss = \int \mathrm{d}k\ k^2 F \tag{5}$$

are to a large extent determined by the second-order spectrum.

It is straightforward to obtain the behaviour of $F(k)$ for large k by taking the appropriate limit of (3),

$$\lim_{k\to\infty} F(k) = \frac{1}{8}\frac{\alpha_p^2}{k k_0^2}, \tag{6}$$

which shows that $F(k)$ behaves like $1/k$; hence parameters such as the wave variance and the mean-square slope really diverge.

The divergence of the expansion in small wave slope has been made plausible in the past by several researchers. The expansion is a small-amplitude development around zero-mean surface. While this may be appropriate for the large-scale waves, small-scale waves are riding on the long waves. Hence for these small waves the domain is not bounded by a zero-mean surface but has a large-scale variation determined by the long waves. This will affect the solution of the potential equation for the short waves and hence will affect the spectrum of the short waves. Others would argue that the divergence of the expansion for high wavenumbers suggests that these short waves become very nonlinear and hence very steep, resulting in micro-scale wave breaking, which would limit energy levels at the high wavenumbers.

However, it turns out that the result of Barrick & Weber (1977) is most likely flawed. This was pointed out for the first time by Creamer *et al.* (1989) who considered improved representations of ocean surface waves using Lie and canonical transformations and applied their work to the determination of the second-order spectrum. Surprisingly, they found the following instead of (1):

$$F(k) = E(k) + \frac{1}{2}k^2 \int_{k/2}^{\infty} \mathrm{d}k'\; E(k')E(|k-k'|) - k^2 E(k) \int_0^{\infty} \mathrm{d}k'\; E(k'). \tag{7}$$

The additional, quasi-linear term was explained by noting that Barrick & Weber (1977) did not include contributions from the product of the first- and third-order surface elevation η, since their second-order spectrum was entirely determined by the second-order surface elevation. It is immediately evident that the additional term cancels the singular behaviour of the first term, as for a Phillips spectrum the extra term equals $-(1/8)\alpha_p^2/kk_0^2$. It is therefore important to include the extra quasi-linear term. In fact, for large wavenumbers one finds from (7) for the Phillips spectrum

$$F(k) = E(k) + \frac{\alpha_p^2}{8k^3}\left[6\log\left(\frac{k^2}{k_0^2}-1\right) - 7\right]; \tag{8}$$

hence, the second-order spectrum behaves in a similar fashion as the first-order Phillips spectrum. This is also shown in figure 1 in which the quasi-linear term shown in (7) gives a large and important correction to the high-wavenumber tail of the second-order spectrum.

The result of Creamer *et al.* (1989) has important consequences for the theory of ocean waves. I therefore thought it worthwhile to check this result by following a somewhat different path, namely choosing as starting point Zakharov's treatment of surface waves. A key role in this approach is the canonical transformation which separates resonant from non-resonant contributions to the evolution of surface waves. The canonical transformation represents the effects of bound waves, and once this transformation is known it is relatively straightforward to obtain an expression for the second-order spectrum. This will be done for the case of two-dimensional propagation for arbitrary spectra. Applying the result for unidirectional waves in deep water the result of Creamer *et al.* (1989) will be recovered. At the same time, it turns out that the approach is so general that it could also be applied for shallow-water waves.

2. Hamiltonian formulation

Modern ocean-wave theories start from the Hamiltonian formulation of the nonlinear evolution equations of the potential flow of an ideal fluid. Zakharov (1968) discovered that the Hamiltonian is given by the total energy of the fluid, while the appropriate canonical variables are the surface elevation $\eta(\boldsymbol{x},t)$ and the value ψ of the potential ϕ at the surface, $\psi(\boldsymbol{x},t)=\phi(\boldsymbol{x},z=\eta,t)$).

Here, the total energy is given by

$$\mathscr{H} = \frac{1}{2}\int\int_{-D}^{\eta} \mathrm{d}z\,\mathrm{d}\boldsymbol{x}\left((\nabla\phi)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right) + \frac{g}{2}\int \mathrm{d}\boldsymbol{x}\;\eta^2.$$

The boundary conditions at the surface, namely the kinematic boundary condition and Bernoulli's equation, are then equivalent to Hamilton's equations,

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta \mathscr{H}}{\delta \eta}, \tag{9}$$

where $\delta \mathscr{H}/\delta \psi$ is the functional derivative of $\mathscr{H}$ with respect to ψ, etc. Inside the fluid the potential ϕ satisfies Laplace's equation,

$$\nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{10}$$

with boundary conditions

$$\phi(\boldsymbol{x}, z = \eta) = \psi \tag{11}$$

and

$$\frac{\partial \phi(\boldsymbol{x}, z)}{\partial z} = 0, \ z = -D, \tag{12}$$

with D the water depth. If one is able to solve the potential problem, then ϕ may be expressed in term of the canonical variables η and ψ. Then the energy $\mathscr{H}$ may be evaluated in terms of the canonical variables, and the evolution in time of η and ψ follows at once from Hamilton's equations given in (9). This was done by Zakharov (1968), who obtained the deterministic evolution equations for deep-water waves by solving the potential problem ((10)–(12)) in an iterative fashion for small steepness ϵ. In addition, the Fourier transforms of η and ψ were introduced, for example

$$\eta = \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k} \ \hat{\eta}(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{13}$$

where $\hat{\eta}$ and similarly $\hat{\psi}$ are the Fourier transforms of η and ψ. Here, $\boldsymbol{k}$ is the wavenumber vector and k its absolute value.

In order to proceed, introduce

$$T_0 = \tanh kD$$

and the linear dispersion relation for surface gravity waves

$$\omega^2 = gkT_0. \tag{14}$$

In waters of arbitrary depth we have the following relation between the Fourier transform of η and ψ and the action variable $A(\boldsymbol{k}, t)$:

$$\hat{\eta} = \sqrt{\frac{\omega}{2g}} (A(\boldsymbol{k}) + A^*(-\boldsymbol{k})), \ \hat{\psi} = -\mathrm{i}\sqrt{\frac{g}{2\omega}} (A(\boldsymbol{k}) - A^*(-\boldsymbol{k})). \tag{15}$$

In terms of the action variable the energy of the fluid becomes to fourth order in amplitude,

$$\begin{aligned} \mathscr{H} = & \int \mathrm{d}\boldsymbol{k}_1 \omega_1 A_1 A_1^* + \int \mathrm{d}\boldsymbol{k}_{1,2,3} \delta_{1-2-3} V^{(-)}_{1,2,3} [A_1^* A_2 A_3 + \text{c.c.}] \\ & + \frac{1}{3} \int \mathrm{d}\boldsymbol{k}_{1,2,3} \delta_{1+2+3} V^{(+)}_{1,2,3} [A_1 A_2 A_3 + \text{c.c.}] \\ & + \int \mathrm{d}\boldsymbol{k}_{1,2,3,4} \delta_{1-2-3-4} W^{(1)}_{1,2,3,4} [A_1^* A_2 A_3 A_4 + \text{c.c.}] \end{aligned}$$

$$+\frac{1}{2}\int \mathrm{d}\boldsymbol{k}_{1,2,3,4}\delta_{1+2-3-4}W^{(2)}_{1,2,3,4}A_1^*A_2^*A_3A_4$$
$$+\frac{1}{4}\int \mathrm{d}\boldsymbol{k}_{1,2,3,4}\delta_{1+2+3+4}W^{(4)}_{1,2,3,4}[A_1^*A_2^*A_3^*A_4^*+\text{c.c.}], \quad (16)$$

where $V^{()}$ and $W^{()}$ are complicated expressions of ω and $\boldsymbol{k}$ that were given by Krasitskii (1994). For convenience all relevant interaction coefficients are also recorded in the Appendix. Here, I have followed a minimalist approach to notation: $A_1=A(\boldsymbol{k}_1,t)$, $\omega_1=\omega(\boldsymbol{k}_1)$, $\mathrm{d}\boldsymbol{k}_{1,2,3}=\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3$, $\delta_{1-2-3}=\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)$, $V^{(-)}_{1,2,3}=V^{(-)}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$, etc. Only when there is a possibility for confusion we shall use a more explicit notation. For example, when introducing two-time-level statistics I will write $\langle a_1(t_1)a_2^*(t_1)\rangle$, where the index '2' refers to the amplitude at wavenumber $\boldsymbol{k}_2$ while the argument refers to time t_1. Finally, as long as there is no confusion we shall use the same symbol for the various forms of the wave spectrum; namely we use F for the complete spectrum (which includes the second-order corrections). The distinction should be clear from their arguments; $F(\boldsymbol{k})$, $F(\omega,\theta)$ and $F(\omega)$ denote wavenumber, directional-frequency and the angular-frequency spectra, respectively. The symbol E will be used for the first-order spectrum.

The evolution equation for A now follows from Hamilton's equation $\partial A/\partial t=-\mathrm{i}\delta\mathscr{H}/\delta A^*$, and evaluation of the functional derivative of the full expression for $\mathscr{H}$ with respect to A^* gives

$$\begin{aligned}\frac{\partial}{\partial t}A_1+\mathrm{i}\omega_1A_1 = {}& -\mathrm{i}\int \mathrm{d}\boldsymbol{k}_{2,3}\Big\{V^{(-)}_{1,2,3}A_2A_3\delta_{1-2-3}+2V^{(-)}_{3,2,1}A_2^*A_3\delta_{1+2-3}\\ &+V^{(+)}_{1,2,3}A_2^*A_3^*\delta_{1+2+3}\Big\}-\mathrm{i}\int \mathrm{d}\boldsymbol{k}_{2,3,4}\Big\{W^{(1)}_{1,2,3,4}A_2A_3A_4\delta_{1-2-3-4}\\ &+W^{(2)}_{1,2,3,4}A_2^*A_3A_4\delta_{1+2-3-4}+3W^{(1)}_{4,3,2,1}A_2^*A_3^*A_4\delta_{1+2+3-4}\\ &+W^{(4)}_{1,2,3,4}A_2^*A_3^*A_4^*\delta_{1+2+3+4}\Big\}. \end{aligned} \quad (17)$$

Equation (17) is the basic evolution equation of weakly nonlinear gravity waves, and it includes the relevant amplitude effects up to third order.

A great simplification of the expression for the energy is achieved by introducing a canonical transformation $A=A(a,a^*)$ that eliminates the contribution of the non-resonant second- and third-order terms as much as possible. The first few terms are given by

$$\begin{aligned}A_1 = a_1 &+ \int \mathrm{d}\boldsymbol{k}_{2,3}\Big\{A^{(1)}_{1,2,3}a_2a_3\delta_{1-2-3}+A^{(2)}_{1,2,3}a_2^*a_3\delta_{1+2-3}\\ &+A^{(3)}_{1,2,3}a_2^*a_3^*\delta_{1+2+3}\Big\}+\int \mathrm{d}\boldsymbol{k}_{2,3,4}\Big\{B^{(1)}_{1,2,3,4}a_2a_3a_4\delta_{1-2-3-4}\\ &+B^{(2)}_{1,2,3,4}a_2^*a_3a_4\delta_{1+2-3-4}+B^{(3)}_{1,2,3,4}a_2^*a_3^*a_4\delta_{1+2+3-4}\\ &+B^{(4)}_{1,2,3,4}a_2^*a_3^*a_4^*\delta_{1+2+3+4}\Big\}\ldots . \end{aligned} \quad (18)$$

The unknowns $A^{()}$ and $B^{()}$ are obtained by systematically removing the non-resonant third- and fourth-order contributions to the wave energy and insisting that the form of the energy remains symmetric. These expressions are quite involved and have been given by Krasitskii (1990, 1994) for example. The derivation of these coefficients is given in the Appendix, and here, we only give the transfer coefficient for the quadratic terms explicitly. They read

$$A^{(1)}_{1,2,3} = -\frac{V^{(-)}_{1,2,3}}{\omega_1 - \omega_2 - \omega_3}, \; A^{(2)}_{1,2,3} = -2\frac{V^{(-)}_{3,2,1}}{\omega_1 + \omega_2 - \omega_3}, \; A^{(3)}_{1,2,3} = -\frac{V^{(+)}_{1,2,3}}{\omega_1 + \omega_2 + \omega_3},$$

and they show that in the absence of resonant three-wave interactions the transformation $A = A(a, a^*)$ is indeed non-singular.

Elimination of the variable A in favour of the new action variable a results in a great simplification of the wave energy $\mathscr{H}$ (see (16)). It becomes

$$\mathscr{H} = \int \mathrm{d}\boldsymbol{k}_1 \omega_1 a_1^* a_1 + \frac{1}{2} \int \mathrm{d}\boldsymbol{k}_{1,2,3,4} T_{1,2,3,4} a_1^* a_2^* a_3 a_4 \delta_{1+2-3-4},$$

where the interaction coefficient $T_{1,2,3,4}$ is given by Krasitskii (1990, 1994) and in § A 1. The interaction coefficient enjoys a number of symmetry conditions, of which the most important one is $T_{1,2,3,4} = T_{3,4,1,2}$ because this condition implies that $\mathscr{H}$ is conserved. In terms of the new action variable a, Hamilton's equation becomes $\partial a/\partial t = -\mathrm{i}\delta\mathscr{H}/\delta a^*$ or

$$\frac{\partial a_1}{\partial t} + \mathrm{i}\omega_1 a_1 = -\mathrm{i} \int \mathrm{d}\boldsymbol{k}_{2,3,4} T_{1,2,3,4} a_2^* a_3 a_4 \delta_{1+2-3-4}, \tag{19}$$

which is known as the Zakharov equation. Clearly, by removing the non-resonant terms, a considerable simplification of the form of the evolution equation describing four-wave processes has been achieved. As a consequence of the canonical transformation the interaction coefficient T now represents two types of four-wave processes. The first type is called the direct interaction and involves the interaction of four free waves (which obey the linear dispersion relation); in the interaction coefficient this process has the weight $W^{(2)}_{1,2,3,4}$. The second type is called a virtual-state interaction because two free waves generate a virtual state consisting of bound waves, which then decays into a different set of free waves. In the interaction coefficient this process is represented by products of the second-order interaction coefficients $V^{\pm}_{1,2,3}$. For narrowband waves in deep water these two processes can be shown to have equal weight.

The Zakharov equation has been used in the past as a starting point for the stability analysis of ocean waves. In addition, it is the appropriate starting point to obtain the Hasselmann equation (see e.g. Janssen 2004) which describes the evolution of the action-density spectrum of an ensemble of surface gravity waves owing to (quasi-)resonant four-wave interactions. The Hasselmann equation forms the cornerstone of present-day wave-forecasting systems. However, strictly speaking one still needs to apply the canonical transformation (18) in order to obtain the surface elevation and the associated wave-variance spectrum. This is the main subject of the present paper. Therefore, the evolution of the free-wave action variable follows from the Zakharov equation, and by applying the canonical transformation (18) the nonlinear corrections to the surface elevation and the wave-variance spectrum may be obtained at every instant. In other words a diagnostic relation will be obtained, which immediately will give the changes in the surface-elevation spectrum due to second harmonics, infra-gravity waves and, in case of the frequency spectrum, the Stokes-frequency correction. Noting that the integral over the surface-elevation spectrum measures the potential energy of the system, it can be shown analytically that for deep-water waves the spectrum is changed in such a way that the total wave variance (hence potential energy) is conserved. By excluding the contributions to the wave spectrum at zero wavenumber we can numerically show that also in shallow water the total wave variance is conserved by the diagnostic relation. It is expected that the conservation of wave variance by the canonical transformation is related to the property of this transformation to ensure that the Zakharov equation is Hamiltonian. However, such a direct connection has not been established yet but deserves further work.

3. Second-order spectrum

The main purpose of this section is to derive a general expression for the wavenumber–angular-frequency spectrum in terms of the interaction coefficients $A^{(i)}(i=1,3)$ and $B^{(i)}(i=1,4)$ that appear in the canonical transformation and the nonlinear interaction coefficient T. Then, from the so-called marginal distribution laws the wavenumber and frequency spectrum are obtained. The main result is that for given free-wave spectrum, which follows from the solution of the energy-balance equation, the canonical transformation provides us with a mapping that immediately gives the appropriate nonlinear low-frequency/wavenumber part of the spectrum and the contributions by second harmonics. This is illustrated by some examples from surface gravity waves in deep water and in water of intermediate depth ($kD \simeq 1$). Compared to the result of Barrick & Weber (1977) two new features are discovered. In agreement with Creamer *et al.* (1989) a quasi-linear term is found, which removes the high-wavenumber catastrophe. In addition, for frequency spectra it is found that the Stokes nonlinear frequency correction contributes to the second-order spectrum.

3.1. *Wavenumber–frequency spectrum*

The purpose of this section is to derive a general expression for wavenumber–frequency spectrum correct to second order. In order to do so we begin by considering the two-point correlation function

$$\rho(\boldsymbol{\xi}, \tau) = \langle \eta(\boldsymbol{x}+\boldsymbol{\xi}, t+\tau)\eta(\boldsymbol{x}, \tau)\rangle,$$

where the angle brackets denote an ensemble average. The wavenumber–frequency spectrum $F(\boldsymbol{k}, \Omega)$ then follows immediately by Fourier transformation in space and time of ρ, i.e.

$$F(\boldsymbol{k}, \Omega) = \frac{1}{8\pi^3}\int \mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\tau\;\rho(\boldsymbol{\xi}, \tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\xi}-\Omega\tau)}. \tag{20}$$

Here, $\boldsymbol{k}$ and angular frequency Ω cover the whole real domain. Note that from the reality of η and the homogeneity of the wave field it follows that the wavenumber–frequency spectrum enjoys the properties $F(\boldsymbol{k}, \Omega) = F^*(\boldsymbol{k}, \Omega) = F(-\boldsymbol{k}, -\Omega)$.

Once the wavenumber–frequency spectrum is known the wavenumber spectrum $F(\boldsymbol{k})$ and the frequency spectrum $F(\Omega)$ follow from the marginal distribution laws:

$$F(\boldsymbol{k}) = \int \mathrm{d}\Omega\;F(\boldsymbol{k}, \Omega);\;\; F(\Omega) = \int \mathrm{d}\boldsymbol{k}\;F(\boldsymbol{k}, \Omega). \tag{21}$$

These marginal distribution laws follow in a straightforward fashion from the definition of the wavenumber–frequency spectrum. For example, the wavenumber spectrum can be obtained by integrating (20) over angular frequency and realizing that the resulting integral over Ω is a δ function in τ space, i.e.

$$\int \mathrm{d}\Omega\;F(\boldsymbol{k}, \Omega) = \frac{1}{8\pi^3}\int \mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\tau\;\rho(\boldsymbol{\xi}, \tau)\int \mathrm{d}\Omega\;\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{\xi}-\Omega\tau)} = \frac{1}{4\pi^2}\int \mathrm{d}\boldsymbol{\xi}\;\rho(\boldsymbol{\xi}, 0)\;\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\xi}} \doteq F(\boldsymbol{k}),$$

and the last equality follows because the wavenumber spectrum is just the Fourier transform of the spatial correlation function. In a similar fashion the relation for the frequency spectrum may be established.

Evaluation of the spatial aspects of the two-point correlation function is fairly straightforward, since in the expression of the surface elevation in (13) we have adopted a Fourier representation in space. Unfortunately, the time aspects of $\rho(\boldsymbol{\xi}, \tau)$ are more complicated, as the action variable $a(\boldsymbol{k}, t)$ obeys the Zakharov equation which is nonlinear. Only when it can be argued that, for example for small wave steepness, the nonlinear term in the Zakharov equation can be neglected, it is straightforward to treat the time aspects of the correlation function as well because the action variable then executes a simple oscillation with the angular frequency of linear surface gravity waves. The latter approach is justified for small wave steepness when one is interested in the lowest order expression of the wavenumber–frequency spectrum (see e.g. Komen *et al.* 1994). Here, we are interested in the second-order spectrum, which is of the order of the square of the lowest order spectrum. The nonlinear term in the Zakharov equation, which gives for example the Stokes-frequency correction for a single wavetrain, is of the order of the amplitude to the third power, and it will be shown that this will give rise to a contribution to the second-order frequency spectrum which is of the same order of magnitude as the generation of second harmonics and the low-frequency set-down.

The relation between two-point correlation function and Fourier amplitude can be established in the following manner. Substitute the Fourier expansion of η into spatial correlation function ρ, and use reality of η ($\hat{\eta}(\boldsymbol{k}) = \hat{\eta}^*(-\boldsymbol{k})$) to establish

$$\rho(\boldsymbol{\xi}, \tau) = \left\langle \int \mathrm{d}\boldsymbol{k}_1 \mathrm{d}\boldsymbol{k}_2 \hat{\eta}(\boldsymbol{k}_1, t_1) \hat{\eta}^*(\boldsymbol{k}_2, t_2) \mathrm{e}^{\mathrm{i}[\boldsymbol{k}_1\cdot\boldsymbol{x} - \boldsymbol{k}_2\cdot(\boldsymbol{x}+\boldsymbol{\xi})]} \right\rangle,$$

where $t_1 = t$ and $t_2 = t + \tau$. For a homogeneous sea,

$$\langle \hat{\eta}(\boldsymbol{k}_1, t_1) \hat{\eta}^*(\boldsymbol{k}_2, t_2) \rangle = R(\boldsymbol{k}_1, \tau) \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2), \tag{22}$$

the correlation function becomes

$$\rho(\boldsymbol{\xi}, \tau) = \int \mathrm{d}\boldsymbol{k}_1 R(\boldsymbol{k}_1, \tau) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{\xi}}.$$

This is then substituted in the expression for the wave spectrum, giving

$$F(\boldsymbol{k}, \Omega) = \frac{1}{2\pi} \int \mathrm{d}\tau \; R(\boldsymbol{k}, \tau) \mathrm{e}^{-\mathrm{i}\Omega\tau}, \tag{23}$$

and further reduction can only be achieved once the time evolution of $R(\boldsymbol{k}, \tau)$ is known.

Clearly, in order to obtain the wavenumber–frequency spectrum evaluation of the second moment $\langle \hat{\eta}(\boldsymbol{k}_1, t_1) \hat{\eta}^*(\boldsymbol{k}_2, t_2) \rangle$ is required. Thus we need the surface elevation in terms of the action variable A (15), and we need the canonical transformation (18). Writing

$$\hat{\eta}_1 = f_1 \left(A(\boldsymbol{k}_1) + A^*(-\boldsymbol{k}_1) \right), \; f_1 = \left(\frac{\omega_1}{2g} \right)^{\frac{1}{2}}, \tag{24}$$

the second moment becomes

$$\langle \hat{\eta}_1(t_1) \hat{\eta}_2^*(t_2) \rangle = f_1 f_2 \langle A_1(t_1) A_2^*(t_2) + A_{-1}^*(t_1) A_{-2}(t_2) + A_1(t_1) A_{-2}(t_2) + A_{-1}^*(t_1) A_2^*(t_2) \rangle.$$

In order to make progress in the evaluation of the second moment, we will make some additional assumptions on the one-time-level statistics of the 'free-wave' action variable a, which are consistent with the Zakharov equation (19). Two-time-level statistics are obtained from the dynamical evolution equation for a directly. First, we assume weakly nonlinear waves; hence $A = O(\epsilon)$, where ϵ is a small parameter of the size of the wave steepness. Since we are interested in the second-order spectrum an answer up to $O(\epsilon^4)$ is required. Second, it is assumed that the action variable a follows the statistics of a homogeneous, stationary field with zero mean value $\langle a_1 \rangle$. Therefore, we introduce the action-density spectrum $N(\boldsymbol{k})$ according to

$$\langle a_1(t_1) a_2(t_1)^* \rangle = N_1 \delta_{1-2}, \tag{25}$$

while $\langle a_1 a_2 \rangle$ vanishes. Because of the cubic nonlinearity in the Zakharov equation the third moment is small, $\langle a_1 a_2 a_3 \rangle = O(\epsilon^5)$, while the fourth moment becomes

$$\langle a_1(t_1) a_2(t_1) a_3(t_1)^* a_4^*(t_1) \rangle = N_1 N_2 \left(\delta_{1-3}\delta_{2-4} + \delta_{1-4}\delta_{2-3} \right) + O(\epsilon^6). \tag{26}$$

The $O(\epsilon^6)$ term is an estimate of the fourth-order cumulant. However, as shown in Janssen (2003), under the exceptional circumstances that freak waves are present, the fourth-order term becomes significantly larger than the present estimate. Strictly speaking, the fourth-order cumulant is, through its dependence on the resonance function, also inversely proportional to the width of the wave spectrum. Hence, wave spectra should be sufficiently wide, or in other words, the so-called Benjamin–Feir index should be sufficiently small. This is most of the time a valid assumption. The exception is, of course, when one is interested in parameters such as excess kurtosis as this quantity is given by an integral over the sixth cumulant. Therefore, for the kurtosis calculation performed in § 5 deviations of the probability density function (p.d.f.) due to the nonlinear dynamics of the Zakharov will be taken into account. The action variable A is now expressed in terms of the free-wave action variable using the canonical transformation (18). For convenience we write (18) in the form

$$A = \epsilon a + \epsilon^2 b(a, a^*) + \epsilon^3 c(a, a^*), \tag{27}$$

where we identify b with the quadratic part of (18) and c with the cubic part of the transformation. Now in shallow water Janssen & Onorato (2007) have shown that there is a wave-induced mean sea level which is generated by the quadratic part of the canonical transformation. In other words, while $\langle a \rangle$ and $\langle c \rangle$ vanish this is not the case for $\langle b \rangle$. However, normally, in agreement with experimental practice, the variance is determined for a process that has zero mean; so for this reason the mean value $\langle b \rangle = \bar{b}\delta_1$ is subtracted from b.

One could contemplate to remove the average level from each member of the ensemble separately, and this will give different results for the wave spectrum and higher-order moments of the p.d.f. because the mean-sea-level correction is nonlinear in wave amplitude. However, this is not in agreement with experimental practice, as one intends to make observations which are representative for the area of interest. For example, if one derives frequency spectra from time series (after subtracting the mean elevation), then these time series need to be sufficiently long in order to be able to compare with the theoretical ensemble averages. A small segment of this time series may be regarded as a certain member of the ensemble, and depending on the number and the strength of the wave groups each segment will have a mean elevation which in general will differ from the mean level over the whole time series. In other words, correcting the signal for the mean elevation per segment would remove an interesting low-frequency signal. As only the mean level over the whole time series is regarded as representative for the sea state we shall subtract the ensemble-average elevation from the elevation signal. As a consequence, we consider instead of (27)

$$A = \epsilon a + \epsilon^2 \tilde{b}(a, a^*) + \epsilon^3 c(a, a^*), \tag{28}$$

with $\tilde{b}_1 = b_1 - \bar{b}_1\delta_1$. As a result, A in (28) has now a zero mean value, and as a matter of fact many terms will cancel in the subsequent calculations. Note that explicitly one finds the following for $\bar{b}$:

$$\bar{b}_1 = \lim_{\boldsymbol{k}_1 \to 0} \int \mathrm{d}\boldsymbol{k}_2 N_2 A^{(2)}_{1,2,2}.$$

Now substitute (28) in the expression for the second moments; then up to fourth order in ϵ one finds

$$\begin{aligned} \langle \hat{\eta}_1(t_1)\hat{\eta}_2^*(t_2)\rangle = f_1 f_2 \{ &\epsilon^2 \langle a_1 a_2^* \rangle + \epsilon^4 \left(\langle \tilde{b}_1 \tilde{b}_2^* \rangle + \langle a_1 c_2^* \rangle + \langle c_1 a_2^* \rangle \right. \\ &\left. + \langle a_1 c_{-2} \rangle + \langle c_1 a_{-2} \rangle + \langle \tilde{b}_1 \tilde{b}_{-2} \rangle \right) \} + \text{c.c.}\,(1 \leftrightarrow -2)\,. \end{aligned} \tag{29}$$

where for brevity $a_1 = a(\boldsymbol{k}_1, t_1)$. The second moment consists of two groups of terms, namely a term proportional to ϵ^2, which will give in lowest order the free-wave spectrum, and the rest of the terms, which, being $O(\epsilon^4)$, contribute to the second-order spectrum. However, the former term, being the dominant one, will also give rise to a contribution to the second-order spectrum, as the free-wave action variable a obeys the nonlinear Zakharov equation.

3.1.1. *First-order spectrum and Stokes-frequency correction*

In this section we are going to evaluate the second moment $g_2 = \langle a_1(t_1) a_2^*(t_2) \rangle$ and in particular its dependence on the timescale $\tau = t_2 - t_1$. The τ dependence of $g_2(\tau)$ is obtained from the Zakharov equation (19), where it is noted that g_2 satisfies according to (25) the initial condition $g_2(\tau = 0) = N_1 \delta_{1-2}$. Evaluating the first τ derivative of g_2 one finds

$$\frac{\partial}{\partial \tau} g_2 = \mathrm{i}\omega_2 g_2 + \mathrm{i} \int \mathrm{d}\boldsymbol{k}_{3,4,5} \langle a_1(t_1) a_3(t_2) a_4^*(t_2) a_5^*(t_2) \rangle \delta_{2+3-4-5}.$$

The evolution equation for g_2 is solved by means of the multiple timescale technique. Thus, one introduces the fast timescale $\tau_0 = \tau$ and the slow timescale $\tau_2 = \epsilon^2 \tau$, together with an expansion of g_2 in terms of the small parameter ϵ^2: $g_2 = \epsilon^2 g_2^{(2)} + \epsilon^4 g_2^{(4)} + \cdots$. In the lowest order one then finds

$$\left(\frac{\partial}{\partial \tau_0} - \mathrm{i}\omega_2 \right) g_2^{(2)} = 0,$$

with solution

$$g_2^{(2)} = G_1(\tau_2)\delta_{1-2} \mathrm{e}^{\mathrm{i}\omega_1 \tau_0}, \tag{30}$$

where $G_1(\tau_2) = G(\boldsymbol{k}_1, \tau_2)$ is still a function of the slow timescale τ_2. The second-order equation becomes

$$\left(\frac{\partial}{\partial \tau_0} - \mathrm{i}\omega_2 \right) g_2^{(4)} = -\frac{\partial}{\partial \tau_2} g_2^{(2)} + S_2,$$

and using the closure assumption

$$\langle a_1(t_1) a_3(t_2) a_4^*(t_2) a_5^*(t_2) \rangle = \epsilon^4 G_1 G_3 \exp(\mathrm{i}\omega_1 \tau_0) \{ \delta_{1-4}\delta_{3-5} + \delta_{1-5}\delta_{3-4} \}$$

the source function S_2 becomes

$$S_2 = 2\mathrm{i}G_1 \mathrm{e}^{\mathrm{i}\omega_1\tau_0} \int \mathrm{d}\boldsymbol{k}_3 T_{1,3,3,1} G_3.$$

Removal of secularity in the second-order equation then gives the slow-time evolution of $G(\tau_2)$,

$$\frac{\partial}{\partial \tau_2} G_1 = 2\mathrm{i}G_1 \int \mathrm{d}\boldsymbol{k}_3 T_{1,3,3,1} G_3,$$

which is all that is needed to evaluate the second-order corrections related to the Stokes-frequency correction.

Returning now to the wavenumber–frequency spectrum (23) we use (30) in (29) to obtain

$$F(\boldsymbol{k}, \Omega) = \frac{f^2(k)}{2\pi} \int \mathrm{d}\tau \, \left\{ G(\boldsymbol{k}, \tau_2) \mathrm{e}^{\mathrm{i}(\omega_1 - \Omega)\tau} + G^*(-\boldsymbol{k}, \tau_2) \mathrm{e}^{-\mathrm{i}(\omega_1 + \Omega)\tau} \right\}.$$

Since G is a slowly varying function of time, it is possible to give an approximate expression for the wavenumber–frequency spectrum by means of partial integration. Alternatively, one may perform a Taylor expansion of $G(\tau)$ for small time. The result is

$$F(\boldsymbol{k}, \Omega) \simeq f^2(k) \left[G(\boldsymbol{k}, 0)\delta(\Omega - \omega(\boldsymbol{k})) + \mathrm{i}\frac{\partial G(\boldsymbol{k}, 0)}{\partial \tau_2} \delta'(\Omega - \omega(\boldsymbol{k})) \right] \\ + \text{c.c.}\ (\boldsymbol{k} \to -\boldsymbol{k}, \Omega \to -\Omega).$$

Making use of the evolution equation for G and the initial condition $G(\tau = 0) = N$ the eventual result is

$$F(\boldsymbol{k}, \Omega) = F_{L+S}(\boldsymbol{k}, \Omega) + (\boldsymbol{k} \to -\boldsymbol{k}, \Omega \to -\Omega), \tag{31}$$

where

$$F_{L+S}(\boldsymbol{k}, \Omega) = \frac{1}{2} E_0 \delta(\Omega - \omega(\boldsymbol{k})) - \frac{1}{2} E_0 \delta'(\Omega - \omega(\boldsymbol{k})) \int \mathrm{d}\boldsymbol{k}_1 \, \hat{T}_{0,1,1,0} E_1,$$

with $\hat{T}_{0,1,1,0} = T_{0,1,1,0}/f_1^2$ and E being the lowest order surface-elevation spectrum,

$$E(\boldsymbol{k}) = \frac{\omega N(\boldsymbol{k})}{g}. \tag{32}$$

The first term in (31), proportional to a δ function, corresponds to the familiar expression for the wavenumber–angular-frequency spectrum of linear ocean waves (cf. Komen *et al.* 1994), while the term proportional to the derivative of the δ function represents a correction due to the Stokes frequency. The latter term is of the order of the square of the wave spectrum and is formally as important as the contributions of the bound waves to the wave spectrum.

3.1.2. *Nonlinear and quasi-linear corrections*

Continuing with the evaluation of the second moment of the surface elevation we are now going to determine the higher-order contributions that are $O(\epsilon^4)$. Since these contributions are of higher order it is sufficient to use the time evolution of the action variables according to linear theory (cf. (30)). The ensemble averages involving a, b and c may be further evaluated by using the quadratic and cubic parts of the canonical transformation. Note that although $\langle a_1 a_2 \rangle$ vanishes this is not the case for correlations such as $\langle a_1 c_{-2} \rangle$ because c_{-2} contains a cubic term which correlates with

a_1. In this fashion one finds

$$\langle a_1 c_{-2} + c_1 a_{-2} \rangle = 2\delta_{1-2} e^{i\omega_1 \tau} \left\{ N_1 \int d\boldsymbol{k}_2 N_2 B^{(3)}_{-1,1,2,2} + N_{-1} \int d\boldsymbol{k}_2 N_2 B^{(3)}_{1,-1,2,2} \right\},$$

while

$$\langle a_1 c_2^* + c_1 a_2^* \rangle = 4\delta_{1-2} N_1 e^{i\omega_1 \tau} \int d\boldsymbol{k}_2 N_2 B^{(2)}_{1,2,2,1}.$$

Furthermore

$$\langle \tilde{b}_1 \tilde{b}_{-2} \rangle = 2\delta_{1-2} \int d\boldsymbol{k}_{3,4} N_3 N_4 \Big[A^{(1)}_{1,3,4} A^{(3)}_{-1,3,4} \delta_{1-3-4} e^{i(\omega_3+\omega_4)\tau} + A^{(3)}_{1,3,4} A^{(1)}_{-1,3,4} \delta_{1+3+4} e^{-i(\omega_3+\omega_4)\tau} + 2A^{(1)}_{4,3,1} A^{(1)}_{3,4,-1} \delta_{1+3-4} e^{-i(\omega_3-\omega_4)\tau} \Big],$$

while

$$\langle \tilde{b}_1 \tilde{b}_2^* \rangle = 2\delta_{1-2} \int d\boldsymbol{k}_{3,4} N_3 N_4 \Big[A^{(1)}_{1,3,4} A^{(1)}_{1,3,4} \delta_{1-3-4} e^{i(\omega_3+\omega_4)\tau} + A^{(3)}_{1,3,4} A^{(3)}_{1,3,4} \delta_{1+3+4} e^{-i(\omega_3+\omega_4)\tau} + 2A^{(1)}_{4,3,1} A^{(1)}_{4,3,1} \delta_{1+3-4} e^{-i(\omega_3-\omega_4)\tau} \Big].$$

Combining everything together, we obtain the fourth-order contribution to the second moment, and from this one immediately then infers $R(\boldsymbol{k}, \tau)$ introduced in (22). According to (23) the wavenumber–frequency spectrum is the Fourier transform of R with respect to time τ, and as a consequence we find the result

$$F(\boldsymbol{k}_1, \Omega_1) = F_{L+S}(\boldsymbol{k}_1, \Omega_1) + \frac{1}{2} \int d\boldsymbol{k}_{2,3} E_2 E_3 \left\{ \mathscr{A}^2_{2,3} \delta_{1-2-3} \delta(\Omega_1 - \omega_2 - \omega_3) + \mathscr{B}^2_{2,3} \delta_{1+2-3} \delta(\Omega_1 + \omega_2 - \omega_3) + 2\mathscr{C}_{2,2,3,3} \delta_{1-2} \delta(\Omega_1 - \omega_2) \right\} + (\boldsymbol{k}_1 \to -\boldsymbol{k}_1, \Omega_1 \to -\Omega_1), \tag{33}$$

where we have added $F_{L+S}(\boldsymbol{k}_1, \Omega_1)$ from (31), while

$$\mathscr{A}_{2,3} = \frac{f_{2+3}}{f_2 f_3} \left(A^{(1)}_{2+3,2,3} + A^{(3)}_{-2-3,2,3} \right), \quad \mathscr{B}_{2,3} = \frac{1}{2} \frac{f_{2-3}}{f_2 f_3} \left(A^{(2)}_{3-2,2,3} + A^{(2)}_{2-3,3,2} \right) \tag{34}$$

and

$$\mathscr{C}_{0,1,2,3} = \hat{B}^{(2)}_{0,3,2,1} + \hat{B}^{(3)}_{-0,1,2,3} = \frac{f_0}{f_1 f_2 f_3} \left(B^{(2)}_{0,3,2,1} + B^{(3)}_{-0,1,2,3} \right) \tag{35}$$

Here, the transfer coefficients $\mathscr{A}$ and $\mathscr{B}$ have a fairly straightforward physical interpretation, as $\mathscr{A}$ measures the strength of the generation of the sum of two waves and hence measures the strength of the generation of second harmonics, while $\mathscr{B}$ measures the generation of low wavenumbers and hence also measures the generation of the mean sea level induced by the presence of wave groups. Apart from a factor of two these coefficients coincide with the work of Longuet-Higgins (1963) on the second-order corrections to the sea-surface elevation. The coefficient $\mathscr{C}$ measures the correction of the first-order amplitude of the free waves by third-order nonlinearity. The transfer coefficients $\mathscr{A}$ and $\mathscr{B}$ are symmetric in their indices,

$$\mathscr{A}_{2,3} = \mathscr{A}_{3,2}, \quad \mathscr{B}_{2,3} = \mathscr{B}_{3,2},$$

while

$$\mathscr{A}_{2,3} = \mathscr{A}_{-2,-3}, \quad \mathscr{B}_{2,3} = \mathscr{B}_{-2,-3}$$

also holds.

The expression for the spectrum $F(\boldsymbol{k}_1, \Omega_1)$ may be further simplified because the presence of the δ functions allows the evaluation of a number of integrals, but no details will be presented here. It suffices to point out that the nonlinear terms (the ones involving $\mathscr{A}$ and $\mathscr{B}$) in (33) agree with the general result obtained by Barrick & Weber (1977), and furthermore, in the special case of one-dimensional propagation, the nonlinear part of the wavenumber–angular-frequency spectrum is found to agree with the result given by Komen (1980), who corrected some misprints found in Barrick & Weber (1977).

3.2. *The wavenumber spectrum*

According to the marginal distribution law (21) the wavenumber spectrum $F(\boldsymbol{k})$ follows from the integration of the wavenumber–frequency spectrum (33) over angular frequency. The general result is

$$F(\boldsymbol{k}_1) = \frac{1}{2}E_1 + \frac{1}{2}\int \mathrm{d}\boldsymbol{k}_{2,3} E_2 E_3 \left\{ \mathscr{A}^2_{2,3}\delta_{1-2-3} + \mathscr{B}^2_{2,3}\delta_{1+2-3} \right\} + E_1 \int \mathrm{d}\boldsymbol{k}_2 E_2 \mathscr{C}_{1,1,2,2} + \{\boldsymbol{k}_1 \rightarrow -\boldsymbol{k}_1\}, \quad (36)$$

From (36) it is seen that the second-order wavenumber spectrum has a fully-nonlinear and a quasi-linear term only. When the wavenumber–frequency spectrum is integrated over angular frequency the contribution by the Stokes-frequency correction vanishes, as expected, as this term is proportional to the derivative of the δ function with respect to Ω_1. This is in agreement with the expectation, as the wavenumber spectrum, being equal to the Fourier transform of the spatial correlation function $\rho(\boldsymbol{\xi}, 0)$, obviously does not explicitly depend on the time evolution as given by the Zakharov equation. It is emphasized that result (36) is for the 'frozen' surface-elevation spectrum, and therefore the wavenumber spectrum $F(\boldsymbol{k})$ is an even function of wavenumber as can easily be verified.

No systematic study has been undertaken so far to investigate under what conditions the result for the wavenumber spectrum (36) converges. For deep-water waves and for realistic wave spectra it was found, and this will be shown in a moment, that the changes to the first-order spectra were small. The situation is different for shallow-water waves because the interaction coefficients become quite large. For the first-order spectra that have been studied in this paper it appears that the changes remain relatively small for $kD > 1$. In the opposite case one might even obtain negative spectra, which is of course highly undesirable.

Before we discuss a number of special cases, namely the case of a single wavetrain and the one-dimensional case of a continuous spectrum of waves propagating in one direction, we mention that using numerical integration it can be shown that the second-order surface-elevation spectrum as given in (36) has the special property that its variance vanishes when the contribution to the spectrum at zero wavenumber is ignored. This is discussed in more detail when moments of the wavenumber and frequency spectrum are discussed in § 3.3.2.

3.2.1. *Single wavetrain*

In this case the first-order spectrum is given by

$$E(\boldsymbol{k}) = m_0 \delta(\boldsymbol{k} - \boldsymbol{k}_0), \quad (37)$$

where m_0 is the zero moment, and substitution of (37) into (36) gives

$$F(\boldsymbol{k}) = \frac{1}{2}m_0\left[1 + 2m_0\left(\hat{B}^{(2)}_{0,0,0,0} + \hat{B}^{(3)}_{-0,0,0,0}\right)\right]\delta(\boldsymbol{k}-\boldsymbol{k}_0) + \frac{1}{2}\mathscr{A}^2_{0,0}m_0^2\delta(\boldsymbol{k}-2\boldsymbol{k}_0) + (\boldsymbol{k}\leftrightarrow -\boldsymbol{k}). \quad (38)$$

Here, we consider the deep-water case only, while the shallow-water effects are treated in § A 3. For deep-water waves in one dimension the expressions for $B^{(2)}$, $B^{(3)}$ and $A^{(i)}$ are relatively simple coefficients. They become

$$B^{(2)}_{0,0,0,0} = -\frac{1}{2}\frac{k_0^3}{\omega_0},\ B^{(3)}_{-0,0,0,0} = \frac{1}{4}\frac{k_0^3}{\omega_0},$$

while

$$A^{(1)}_{0+0,0,0} = \frac{1}{4}\left(\frac{2g}{\omega_{0+0}}\right)^{\frac{1}{2}}(1+\sqrt{2})\frac{k_0^2}{\omega_0},\ A^{(3)}_{-0-0,0,0} = \frac{1}{4}\left(\frac{2g}{\omega_{0+0}}\right)^{\frac{1}{2}}(1-\sqrt{2})\frac{k_0^2}{\omega_0}.$$

Hence, the coefficients in (38) read

$$\hat{B}^{(2)}_{0,0,0,0} = -k_0^2,\ \hat{B}^{(3)}_{-0,0,0,0} = \frac{k_0^2}{2},\ \mathscr{A}^2_{0,0} = k_0^2, \quad (39)$$

and therefore, from (38) one obtains as positive wavenumber spectrum $F_+(\boldsymbol{k}) = 2F(\boldsymbol{k})$ $(\boldsymbol{k} > 0)$,

$$F_+(\boldsymbol{k}) = m_0\left\{\left(1-k_0^2m_0\right)\delta(\boldsymbol{k}-\boldsymbol{k}_0) + k_0^2m_0\delta(\boldsymbol{k}-2\boldsymbol{k}_0)\right\}. \quad (40)$$

It is immediately evident from the above expression that the canonical transformation gives a second-order correction to the shape of the wave spectrum, which results in an additional second-harmonic peak at $\boldsymbol{k} = 2\boldsymbol{k}_0$, while the energy of the first harmonic at $\boldsymbol{k} = \boldsymbol{k}_0$ also has a correction. In agreement with the energy preserving property of the canonical transformation the wave variance of the total spectrum is, however, unchanged as

$$\int \mathrm{d}\boldsymbol{k}\ F_+(\boldsymbol{k}) = m_0.$$

Therefore, the increase in wave variance due to the presence of the peak at twice the wavenumber $\boldsymbol{k}_0$ is exactly compensated by the second-order correction to the energy of the first harmonic. The latter correction can be traced back to the interaction coefficients $B^{(2)}$ and $B^{(3)}$ (see (38)). In particular, $B^{(2)}$ causes a reduction of the wave variance at the first harmonic (see (39)), and as explained in § A 1 the form of this coefficient has been chosen in such a way that the free-wave action variable a obeys an evolution equation which is Hamiltonian. In § A 3 we derive the wave spectrum of a single wavetrain in a slightly different fashion by writing down the canonical transformation for a single wavetrain (A 14) and by deriving the corresponding expression for the surface elevation. It is then straightforward to obtain the wave spectrum by evaluation of the Fourier transform of the spatial correlation function (cf. (A 19)). The present expression for the single-wave spectrum given in (40) is in perfect agreement with the deep-water version of (A 19) given in § A 3.

In § A 3 it is also pointed out that the usual Stokes expansion for a single wavetrain is not unique. In fact, there is a whole family of solutions that satisfies the Hamilton equations (17). The canonical transformation for the single wavetrain belongs to this family. This transformation is unique, however, because the single mode is regarded as the limit of the continuous case, while the canonical transformation for general wave spectra has to satisfy the additional requirement that the equations of motion remain Hamiltonian.

3.2.2. *Continuous spectrum of waves propagating in one direction*

We now take the case of one-dimensional propagation, and we assume that the waves are propagating in the positive x direction. Therefore,

$$E(k) = \begin{cases} E(k), \ k > 0, \\ 0, \ k < 0. \end{cases}$$

For this choice of lowest order wave spectrum the expression for the wave spectrum (36) may be simplified considerably. The positive wavenumber spectrum becomes

$$F_+(k_1) = E_1 + 2E_1 \int_0^\infty \mathrm{d}k_2 E_2 \left(\hat{B}^{(3)}_{-1,1,2,2} + \hat{B}^{(2)}_{1,2,2,1}\right) + \int_0^{k_1} \mathrm{d}k_2 E_2 E_{1-2} \mathscr{A}^2_{2,1-2} + \int_0^\infty \mathrm{d}k_2 E_2 E_{1+2} \mathscr{B}^2_{2,1+2} + \int_{k_1}^\infty \mathrm{d}k_2 E_2 E_{2-1} \mathscr{B}^2_{2,2-1}. \quad (41)$$

For numerical evaluation of expression (41) one needs to rewrite the convolution integrals, in particular the third and the fifth term of the right-hand side, because the argument $k_1 - k_2$ or $k_2 - k_1$ vanishes in the integration range. When both k_1 and k_2 are large, the integral involves the product of energy at low wavenumbers, which is large, with energy at high wavenumbers, giving very noisy results for the high-wavenumber spectrum (unless one would be able to discretize with very large resolution). In order to avoid noisy results I have transformed the third and the fifth term in such a way that these conditions do not occur. For example in the third term the integration interval is split in two, namely from 0 to $k_1/2$ and from $k_1/2$ to k_1. Next, because the integrals are of the convolution type and $\mathscr{A}$ is symmetric, it is straightforward to show that the second integral equals the first. Furthermore, the fifth integral can be written as an integral over the domain 0 to ∞ by using the transformation $k_2 - k_1 = k_3$. Then, using the symmetry property of $\mathscr{A}$, the result is identical to the fourth integral. As a consequence, (41) becomes

$$F_+(k_1) = E_1 + 2E_1 \int_0^\infty \mathrm{d}k_2 E_2 \left(\hat{B}^{(3)}_{-1,1,2,2} + \hat{B}^{(2)}_{1,2,2,1}\right) + 2\int_0^{k_1/2} \mathrm{d}k_2 E_2 E_{1-2} \mathscr{A}^2_{2,1-2} + 2\int_0^\infty \mathrm{d}k_2 E_2 E_{1+2} \mathscr{B}^2_{2,1+2}. \quad (42)$$

Note that substitution of the single-mode spectrum given in (37) into (42) yields result (40).

In agreement with Creamer *et al.* (1989) the second-order spectrum consists of two contributions, a fully nonlinear contribution (the last two terms of (42)) and a quasi-linear term (the second term of (42)). We will now show that in deep water the fully nonlinear term is in agreement with the result of Barrick & Weber (1977), while the expression for the quasi-linear term agrees with Creamer *et al.* (1989). In order to show this one needs to evaluate the transfer coefficients for the one-dimensional case. Making use of the work of Jackson (1979) and numerical evaluations I have found

$$\mathscr{A}_{1,2} = \frac{s_1 s_2}{2}|k_1 + k_2|, \ \mathscr{B}_{1,2} = -\frac{s_1 s_2}{2}|k_1 - k_2|, \quad (43)$$

where s_1 and s_2 denote the signs of the wavenumbers k_1 and k_2.

Substitution of (43) into the fully nonlinear terms NL then gives

$$NL = \frac{k_1^2}{2}\int_0^{\frac{k_1}{2}} \mathrm{d}k_2 E_2 E_{1-2} + \frac{k_1^2}{2}\int_0^{\infty} \mathrm{d}k_2 E_2 E_{1+2}.$$

The first integral equals the integral with the same argument over the domain $(k_1/2, k_1)$, while the last integral can be rewritten in an integral over the domain (k_1, ∞), and the result becomes

$$NL = \frac{k_1^2}{2}\int_{k_1/2}^{\infty} \mathrm{d}k_2 E_2 E_{|1-2|},$$

which agrees with (2).

Next, the coefficients in the quasi-linear term are evaluated. In one dimension I found (with the help of Miguel Onorato who used Mathematica) the simple expressions

$$\hat{B}^{(2)}_{1,2,2,1} = -\frac{1}{2}k_1^2\left(1+\frac{\omega_2}{\omega_1}\right), \; \hat{B}^{(3)}_{-1,1,2,2} = \frac{1}{2}k_1^2\frac{\omega_2}{\omega_1}, \rightarrow \mathscr{C}_{1,1,2,2} = -\frac{1}{2}k_1^2, \tag{44}$$

and the quasi-linear term QL becomes

$$QL = -k_1^2 E_1 \int_0^{\infty} \mathrm{d}k_2 E_2,$$

which agrees with the result of Creamer *et al.* (1989). The resulting spectrum, correct to second order becomes

$$F_+(k_1) = E_1 + \frac{k_1^2}{2}\int_{k_1/2}^{\infty} \mathrm{d}k_2 E_2 E_{|1-2|} - k_1^2 E_1 \int_0^{\infty} \mathrm{d}k_2 E_2, \tag{45}$$

which is in complete accord with result (7). Hence, it is concluded that the quasi-linear term, evaluated with the formalism developed by Zakharov, plays an important role, as it removes a divergent part from the fully nonlinear term. As a consequence, it seems likely that the Hamiltonian approach of Zakharov combined with the canonical transformation of Krasitskii leads to convergent results. The advantage of this approach over the one by Creamer *et al.* (1989) is that we now immediately have the generalization to two dimensions as well (see (36)).

As a final check of the results we have evaluated numerically the second-order spectrum by using the general expression given in (42). All integrals in this paper will be evaluated with the trapezoid rule on a grid with variable resolution. The wavenumbers are on a logarithmic scale with $\Delta k/k = 0.10$, and the total number of waves is $N = 80$, therefore spanning a wavenumber range $k_{max}/k_{min} = (1+\Delta k/k)^{N-1}$ which is typically a factor of 2000. The result of this integration is shown in figure 1 and coincides with the analytical result labelled with (3), (4) and (7). The second-order spectrum remains indeed small compared to the first-order result. Furthermore, it has been checked that also for the standing-wave case, which has potentially a stronger nonlinearity, the quasi-linear term removes the divergent part of the nonlinear term. In fact, in the latter case one finds that for deep-water waves the second-order spectrum is precisely twice the one in the propagating example (cf. (45)).

3.3. *Angular-frequency spectrum*

In order to obtain the directional-frequency spectrum $F(\Omega, \theta)$, where θ is the propagation direction of the waves, we introduce polar coordinates in wavenumber space so that for example we have for the first-order spectrum

$$E(\boldsymbol{k})\mathrm{d}\boldsymbol{k} = E(k,\theta)k\mathrm{d}k\mathrm{d}\theta = E(\Omega,\theta)\mathrm{d}\Omega\mathrm{d}\theta_1 \rightarrow E(\boldsymbol{k}) = v_g(k)E(\Omega,\theta)/k.$$

According to the marginal distribution law (21) the angular-frequency spectrum follows from the integration of the wavenumber–frequency spectrum over the wave vector $\boldsymbol{k}$. However, our interest is in the directional-frequency spectrum $F(\Omega,\theta)$, and we define it by integrating $F(\boldsymbol{k},\Omega)$ over the absolute wavenumber $k=|\boldsymbol{k}|$ only and by considering positive frequencies only (hence the factor of two):

$$F(\Omega,\theta) = 2\int k\mathrm{d}k\ F(\boldsymbol{k},\Omega),\ \ \Omega > 0. \tag{46}$$

A number of integrations in (46) may be performed because of the presence of three δ functions in the wavenumber–frequency spectrum given in (33), and the directional-frequency spectrum becomes after some straightforward algebraic manipulations

$$\begin{aligned} F(\Omega_1,\theta_1) = E(\Omega_1,\theta_1) &- \frac{\partial}{\partial\Omega_1}\left\{E(\Omega_1,\theta_1)\int \mathrm{d}\Omega_2\mathrm{d}\theta_2\ \hat{T}_{1,2,2,1}E(\Omega_2,\theta_2)\right\} \\ &+ 2\int_0^{\Omega_1/2} \mathrm{d}\Omega_2\mathrm{d}\theta_2\ E(\Omega_1-\Omega_2,\theta_1)E(\Omega_2,\theta_2)\mathscr{A}^2_{1-2,2} \\ &+ 2\int_0^{\infty} \mathrm{d}\Omega_2\mathrm{d}\theta_2\ E(\Omega_1+\Omega_2,\theta_1)E(\Omega_2,\theta_2)\mathscr{B}^2_{1+2,2} \\ &+ 2E(\Omega_1,\theta_1)\int \mathrm{d}\Omega_2\mathrm{d}\theta_2\ E(\Omega_2,\theta_2)\mathscr{C}_{1,1,2,2}. \end{aligned} \tag{47}$$

This is the main result of this section. For given first-order, free-wave spectrum $E(\Omega,\theta)$, (47) gives the second-order corrections to the free-wave spectrum. However, it must be emphasized that all wavenumbers in the above mapping relation should be converted to angular frequencies using the inverse of the dispersion relation (14): For example $k_2 = k(\Omega_2)$, while $k_{1-2} = k(\Omega_1-\Omega_2)$ and $k_{1+2} = k(\Omega_1+\Omega_2)$. Although for deep water the expressions for these wavenumbers can be obtained explicitly, for shallow water this can only be done numerically using an iteration scheme.

It is instructive to compare the result for the frequency-direction spectrum with the one for the wavenumber spectrum given in (36). It is then clear that the fully nonlinear terms and the quasi-linear term in (47) have, regarding their form, a close resemblance to the corresponding terms in the wavenumber spectrum. However, the frequency-direction spectrum has an additional term which is related to a Doppler shift of the frequency by nonlinear effects (the so-called Stokes-frequency correction). Note that this term involves minus the derivative of the first-order frequency spectrum with respect to frequency, and therefore, in deep water in which the Stokes-frequency correction is positive the result will be a shift of the frequency spectrum towards higher frequencies, while in shallow water in which the Stokes-frequency correction is negative the frequency spectrum will be shifted towards lower frequencies. For a detailed discussion of this effect on deep-water single wavetrains see Janssen & Komen (1982).

3.3.1. *Deep-water waves in one dimension*

For one-dimensional deep-water waves it is fairly straightforward to obtain the interaction coefficients (see (43) and (44) for $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$). Furthermore, the interaction

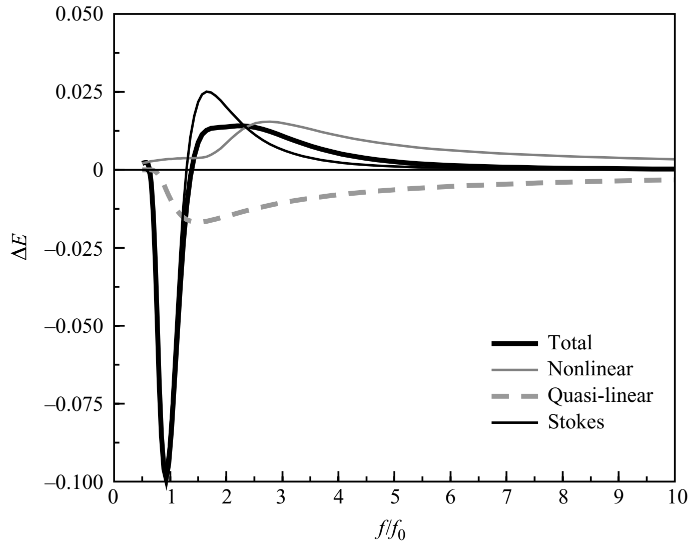

FIGURE 2. The second-order effects on the frequency spectrum as functions of f/f_0. In addition, the effects of the fully nonlinear term, the quasi-linear term and the Stokes-frequency correction are given separately.

coefficient $T_{1,2,2,1}$ is given by the simple expression (Zakharov 1991)

$$T_{1,2,2,1} = \begin{cases} k_1 k_2^2, & k_2 < k_1, \\ k_1^2 k_2, & k_2 > k_1. \end{cases}$$

Substituting all this in (47) the frequency spectrum for unidirectional waves becomes

$$\begin{aligned} F(\Omega_1) = {} & E(\Omega_1) - \frac{2}{g^2}\frac{\partial}{\partial \Omega_1} E(\Omega_1)\left\{ \Omega_1^2 \int_0^{\Omega_1} \mathrm{d}\Omega_2\, \Omega_2^3 E(\Omega_2) + \Omega_1^4 \int_{\Omega_1}^{\infty} \mathrm{d}\Omega_2\, \Omega_2 E(\Omega_2) \right\} \\ & + \frac{1}{2g^2}\left\{ \int_0^{\Omega_1/2} \mathrm{d}\Omega_2\, E(\Omega_1 - \Omega_2)E(\Omega_2)\left[(\Omega_2 - \Omega_1)^2 + \Omega_2^2\right]^2 \right. \\ & \left. + \Omega_1^2 \int_0^{\infty} \mathrm{d}\Omega_2\, E(\Omega_1 + \Omega_2)E(\Omega_2)(\Omega_1 + 2\Omega_2)^2)\right\} - \frac{\Omega_1^4}{g^2} E(\Omega_1) \int_0^{\infty} \mathrm{d}\Omega_2\, E(\Omega_2). \end{aligned} \tag{48}$$

Note that the fully nonlinear contribution to the second-order frequency spectrum is in complete agreement with a result obtained by Komen (1980). Let us study in more detail the angular-frequency spectrum and in particular the consequences of the nonlinear corrections, for the realistic case of a Joint North Sea Wave Project (JONSWAP) spectrum (Hasselmann *et al.* 1973) with the peak frequency $\Omega_0 = 0.5$, the Phillips parameter $\alpha_p = 0.01$ and the overshoot parameter $\gamma = 1$. In figure 2 we show the frequency dependence of the total increment to the first-order JONSWAP spectrum due to second-order effects, and in addition we show increments due to the fully nonlinear term, the quasi-linear term and the Stokes-frequency correction separately as given by (48). The fully nonlinear term is always positive and with increasing frequency shows a sudden increase around twice the peak frequency, while for large frequencies it has an f^{-1} tail. The quasi-linear term is always negative, and it attains its minimum value around $f = 1.5 f_0$. This term also has an f^{-1} tail which,

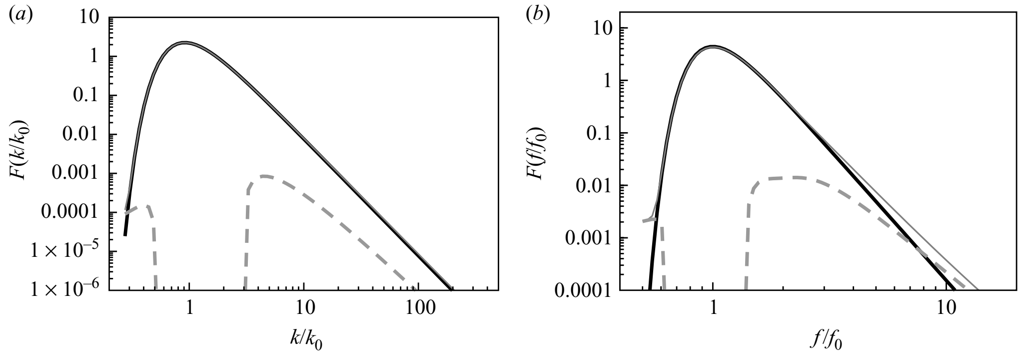

FIGURE 3. Comparison of the wavenumber and frequency spectra including the second-order effects (grey line). For clarity the first-order spectrum (black line) and the second-order contribution (dashed grey line) are shown as well. For deep-water waves the Stokes-frequency correction is hardly visible near the peak of the frequency spectrum, while the second-order effects have a pronounced impact on the high-frequency tail of the wave spectrum. However, the second-order effects on the wavenumber spectrum are not visible.

as will be seen in a moment, cancels the tail of the fully nonlinear term in such a way that in agreement with (49) the sum of the two terms has an f^{-3} behaviour. For deep-water waves the Stokes-frequency correction gives rise to a shift of the wave spectrum towards higher frequencies, and therefore in figure 2 we see a typical negative–positive signature of this term. In the frequency range of $1.2f_0 < f < 2f_0$ the Stokes-frequency correction compensates the effect of the quasi-linear term, while for large frequencies it falls off more rapidly than both the fully nonlinear term and the quasi-linear term. Adding all contributions together it is seen that the main effect is a shift of the low-frequency part of the wave spectrum towards higher frequencies, while at high frequencies there is a small increase in spectral levels. One would conclude from figure 2 that the Stokes-frequency correction plays an important role in the modification of the frequency spectrum, but the main change is near the peak of the first-order spectrum which has most of the variance. As a consequence, for the present example the Stokes-frequency correction only gives a small modification of the first-order spectrum, while the small increments at high frequency give a relatively large modification of the first-order spectrum. This follows from figure 3(*b*) that shows the first-order frequency spectrum, the contribution by second-order effects and the total spectrum. Therefore, as far as the total spectrum is concerned, the main second-order effect is a somewhat fatter high-frequency tail. Only for young, steep windsea (having an overshoot parameter $\gamma \simeq 3$ and a Phillips parameter $\alpha_p \simeq 0.02$) or, as will be evident in the next section, only in the fairly extreme circumstances of shallow water a significant impact of the Stokes-frequency correction on the frequency spectrum is to be found.

The presence of a somewhat fatter high-frequency tail in the frequency spectrum has important consequences; so let us discuss this aspect in more detail. A fit of the high-frequency part of the spectrum from 2 times the peak frequency until 10 times the peak frequency with a power law of the type f^{-m} gives a slope m of about 4. This is intriguing, as this slope has been reported frequently in observational studies (Toba 1973; Kawaii, Okuda & Toba 1977; Mitsuyasu *et al.* 1980; Forristall 1981; Kahma 1981; Donelan, Hamilton & Hui 1985), but later experimental studies suggest that at high frequencies there is a transition from f^{-4} to f^{-5} (e.g. Hara &

Karachintsev 2003). There are also a number of theoretical explanations in favour of an f^{-4} power law. These range from the familiar concept of the Kolmogorov inertial energy cascade caused by the resonant four-wave interactions (Zakharov & Filonenko 1967) to Doppler shifting of short waves by the presence of the orbital motion of the long waves (e.g. Banner 1990); Belcher & Vassilicos (1997) have explained the f^{-4} power law in terms of the dominance of bound waves (associated with sharp crested free gravity waves) over the high-frequency free waves. Our explanation of a fatter high-frequency tail comes closest to the work of Belcher & Vassilicos (1997). In the present approach the occurrence of sharp crested waves is implicit in the choice of the high-frequency tail of the first-order spectrum (a Phillips spectrum), but alternative choices of a first-order spectrum will give rise to a fatter tail as well. Note that we have considered unidirectional waves only, and it would be of interest to study effects of directionality (cf. (47)) on wave-variance levels at high frequencies. This is left for further study.

The presence of an enhanced tail in the high-frequency spectrum is also plainly evident in the following simple example. Hence, for the Phillips spectrum (3), converted to angular frequency space,

$$E(\Omega) = \alpha_p g^2 \Omega^{-5}, \; \Omega > \Omega_0,$$

it is possible to evaluate all integrals in (48) explicitly, but the resulting analytical expression looks much more elaborate than the corresponding one for the wavenumber spectrum (c.f. (3) and (4)); so we will not present these details. It is only mentioned that the second-order corrections to the angular-frequency spectrum play indeed a much more important role than in case of the wavenumber spectrum. To be definite, from the exact solution one may obtain an asymptotic expansion in powers of the square of Ω/Ω_0, valid for large frequencies,

$$F(\Omega) \simeq E(\Omega)\left(1 + \frac{\alpha_p}{2}\frac{\Omega^2}{\Omega_0^2}\right), \; \Omega \gg \Omega_0, \tag{49}$$

which shows that there is a considerable contribution to the frequency spectrum by the bound waves as it scales with Ω^{-3}. In sharp contrast, the contribution of the bound waves to the wavenumber spectrum scales apart from a logarithmic dependence as k^{-3}, which is a similar behaviour as the first-order spectrum (cf. (8)). Therefore, bound waves give rise to a fatter high-frequency tail, while at the same time in the wavenumber domain the contribution of the bound waves is small. This is illustrated in figure 3(a) in which the wavenumber spectrum shows hardly any change in the high-wavenumber tail due to the bound waves, while in figure 3(b) there are visible changes to the frequency spectrum to be noted.

3.3.2. *A remark on moments of the spectrum*

It can be readily verified that the zeroth moment of the second-order spectrum for the case of one-dimensional propagation vanishes. This follows from the numerical evaluations in deep water and also in shallow water when the contributions to the wave spectrum at zero wavenumber are ignored. The question whether this conservation property can be proven in an analytical manner is therefore of interest. For deep-water waves this follows immediately from an integration of the general result for the wavenumber spectrum (36) over wavenumber with the result

$$\langle \eta^2 \rangle = \int \mathrm{d}\boldsymbol{k}_1 E_1 + \int \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2 \mathscr{T}_{1,2} E_1 E_2, \tag{50}$$

where

$$\mathscr{T}_{1,2} = \mathscr{A}^2_{1,2} + \mathscr{B}^2_{1,2} + 2\mathscr{C}_{1,1,2,2}.$$

Upon using the expressions for the interaction coefficients given in (43) and (44) the vanishing of the second integral follows at once, as the transfer coefficient $\mathscr{T}$ is antisymmetric: $\mathscr{T}_{1,2} = -\mathscr{T}_{2,1}$. Hence, even in the presence of bound waves, the wave variance is given by the integral over the first-order spectrum only. A similar proof may be given for the second-order frequency spectrum, while this also follows in a trivial way from the wavenumber–frequency spectrum and the marginal distributions laws (21). Note that I have been unable to obtain a proof of this property of the second-order spectrum for two-dimensional propagation in deep water. However, Sergei Annenkov (2009, private communication), who read a first draft of the present paper, pointed out to me that he managed to prove that the transfer coefficient $\mathscr{T}_{1,2}$ is also antisymmetric in the case of two-dimensional propagation of deep-water waves. To this end he used Maple to express the transfer coefficient in terms of the wavenumber and angular frequency. For shallow-water waves only an analytical proof is available in the case of a single wavetrain. To that end one uses the expression for the spectrum of a single wavetrain given in (A 19) and ignores the contribution at zero wavenumber. Upon using (A 17) the vanishing of the variance of the second-order spectrum follows at once. It should be clear, however, that all other moments of the spectrum are affected by the presence of bound waves. We will discuss this in some detail for the mean-square slope of deep-water waves, as this quantity is relevant in satellite-retrieval algorithms, the albedo of the sea surface and in air–sea interaction studies. It is important to realize that in the presence of bound waves the mean-square slope, *mss*, does not follow from the usual fourth moment of the frequency spectrum. For free waves, obeying the linear dispersion relation $\Omega = \omega(k)$ it can be shown that indeed $\int \mathrm{d}k\, k^2 F(k) = \int \mathrm{d}\Omega\, (\Omega^4/g^2) F(\Omega)$, and hence the fourth moment of the frequency spectrum equals the mean-square slope. However, bound waves do not obey the dispersion relation from linear theory, while, in addition, the frequency spectrum shifts towards higher frequencies because of the Stokes-frequency correction. This is most easily understood by considering the example of a single wavetrain. Substituting the expression for the spectrum of a single wavetrain, i.e.

$$E(k) = m_0 \delta(k - k_0),$$

in (33) one finds for the wavenumber–frequency spectrum

$$F(k,\Omega) = \frac{1}{2} m_0 \left(1 - k_0^2 m_0\right) \delta(k-k_0)\delta(\Omega-\omega_0) - k_0^2 m_0^2 \omega_0 \delta(k-k_0)\delta'(\Omega-\omega_0) + \frac{1}{2} k_0^2 m_0^2 \delta(k-2k_0)\delta(\Omega-2\omega_0) + (k \to -k, \Omega \to -\Omega).$$

Here, the first term combines the linear term and the quasi-linear effect; the second term represents the effect of the Stokes-frequency correction; and the third term gives the generation of second harmonics. The wavenumber spectrum follows immediately from an integration over angular frequency,

$$F(k) = \int \mathrm{d}\Omega\; F(k,\Omega) = m_0 \left(1 - k_0^2 m_0\right) \delta(k-k_0) + k_0^2 m_0^2 \delta(k-2k_0),$$

and hence the mean-square slope becomes

$$mss = \int \mathrm{d}k\; k^2 F(k) = k_0^2 m_0 \left(1 + 3k_0^2 m_0\right).$$

On the other hand, the frequency spectrum follows from the marginal distribution law (46); hence

$$F(\Omega) = m_0\left(1 - k_0^2 m_0\right)\delta(\Omega - \omega_0) - 2k_0^2 m_0^2 \omega_0 \delta'(\Omega - \omega_0) + k_0^2 m_0^2 \delta(\Omega - 2\omega_0),$$

and the fourth moment of the frequency spectrum m_4 becomes

$$m_4 = \int \mathrm{d}\Omega \, \frac{\Omega^4}{g^2} F(\Omega) = k_0^2 m_0 \left(1 + 23 k_0^2 m_0\right).$$

Evidently there is a considerable difference between m_4 and *mss*. There are two reasons for this difference. First, the frequency of the waves is subject to a Doppler shift caused by the Stokes-frequency correction which shifts the frequency spectrum towards higher frequencies. Second, the second harmonic has a frequency $2\omega_0$ and a wavenumber $2k_0$, but according to the fourth moment the wave variance at $2\omega_0$ has a wavenumber $4k_0$ as $k = \omega^2/g = 4\omega_0^2/g$. Hence, for deep-water waves the fourth moment m_4 and the mean-square slope, *mss*, will be different. Returning now to figure 3 in which a comparison of wavenumber and frequency spectra is shown, it is immediately evident that also for a continuous spectrum the fourth moment is larger than the mean-square slope, as due to the nonlinear corrections the level of the high-frequency part of the frequency spectrum has increased. This has important consequences for the estimation of the mean-square slope from frequency spectra as obtained from buoy time series. Assuming that buoys can observe only frequencies below a cutoff frequency, say 0.5 Hz, well-resolved sea states, corresponding to large wave heights, are in particular prone to an overestimation of the mean-square slope. Using a JONSWAP spectrum the overestimation due to the incorrect interpretation of the fourth moment as a proxy for mean-square slope may be determined. For example for a wind speed of $20\,\mathrm{m\,s^{-1}}$ and a wave height of 10 m the mean-square slope may be overestimated by 30 %, while a low-wave-height case only gives an overestimation of 5 %. Therefore, estimates of the mean-square slope from frequency spectra may have considerable errors.

3.3.3. *Shallow-water effects*

Let us apply now the general expression for the directional-frequency spectrum (47) to the case of shallow water. It was already mentioned that in order to evaluate the second-order contribution to the frequency spectrum in waters of finite depth the inverse of the dispersion relation (14) is required. However, in the shallow-water case this inversion cannot be given in an analytical manner; therefore only numerical results will be presented in this section.

The examples that will be discussed here are taken from the *Coastal Engineering Manual* (US Army Corps of Engineers, 2002, chapter 4, p. II-4-16) on surf-zone hydrodynamics. In this manual three examples of wave spectra in shallow water are shown for depths of 3, 1.7 and 1.4 m, but only the first two cases will be considered, as the shallowest example is in the surf zone, where violent breaking occurs, which is not taken into account in the present context. As first-order spectrum we take a JONSWAP spectrum with the peak angular frequency $\Omega_0 = 2.1$, the Phillips parameter $\alpha_p = 0.015$ and the overshoot parameter $\gamma = 7$, with the frequency width $\sigma = 0.07$. For depths D of 3 and 1.7 m the dimensionless depths $k_0 D$ at the peak of the spectrum are 1.65 and 1.00 respectively. For the case in the surf zone with $D = 1.4$ m the dimensionless depth is 0.89 which is beyond the limit of convergence of the present approach (47).

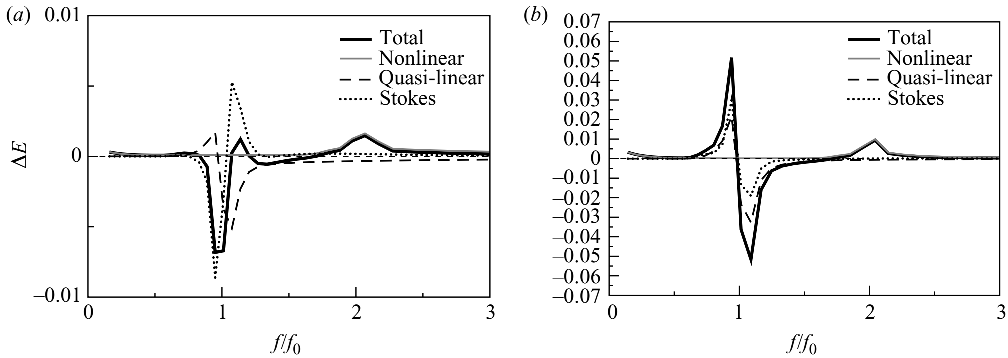

FIGURE 4. The second-order effects on the frequency spectrum as functions of f/f_0. The effects of the fully nonlinear term, the quasi-linear term and the shift by the Stokes-frequency correction are given separately as well. (*a*) The case $D = 3$ m ($k_0D = 1.49$). (*b*) The case $D = 1.7$ m ($k_0D = 1.00$). Note the pronounced difference in the shift due to the Stokes-frequency correction, being positive in (*a*) and negative in (*b*). Also note the change of scale suggesting the sensitive dependence of the second-order spectrum on depth.

Let us study the increments for the cases $D = 3$ m and $D = 1.7$ m using the same first-order spectrum. They are shown in figure 4. First of all note the change of scale by a factor of 5 when going towards shallower water, indicating that indeed the second-order spectrum depends in a sensitive manner on depth. Second, while the increments for the nonlinear and quasi-linear terms are qualitatively similar, the increments due to the Stokes-frequency correction are markedly different. The case of $k_0D = 1.49$ ($D = 3$ m) is similar to the deep-water problem and has a positive frequency shift, while for $k_0D = 1.00$ ($D = 1.7$ m) the frequency shift is negative. This is qualitatively in agreement with the well-known result that for a single wavetrain the Stokes-frequency correction is positive for $kD > 1.363$, while it is negative in the opposite case (Whitham 1974, p. 636; Janssen & Onorato 2007). However, the present case is not quite narrowband, and by trial and error it was found that the transition from positive shift to negative shift occurred at a slightly lower value of the dimensionless depth, namely $k_0D \simeq 1.2$. In contrast with deep-water waves the increments due to the Stokes-frequency correction are now quite significant, and they are visible near the peak of the total wave spectrum. This is illustrated in figure 5 in which for the same first-order spectrum the sum of first-and second-order spectra is shown for the two values of depth. Comparing the first-order spectrum with the total spectrum it is clear that for $D = 3$ m there is hardly any shift of the spectrum, while for the shallower case $D = 1.7$ m there is a definite downshift of the total spectrum, therefore once more supporting the sensitive dependence of the second-order spectrum on depth. In particular, note the rapid increase of the low-frequency infra-gravity wave energy by a factor of 10, while the dimensionless depth only decreases by about 60 %; moreover, the second-harmonic peak appears to be sensitive to depth variations. Finally, the increased high-frequency levels caused by second-order nonlinearity are evident in figure 5. In both cases the high-frequency part of the spectrum follows closely an f^{-4} power law for frequencies larger than 1 Hz. Removing the quasi-linear effect would, just as in the case of deep-water spectra, result in a much more rapid divergence from the first-order spectrum. This is illustrated in figure 6 in which it is clear that without the quasi-linear term higher levels in the high-frequency part of the spectrum are obtained. Observations of the frequency spectrum were kindly

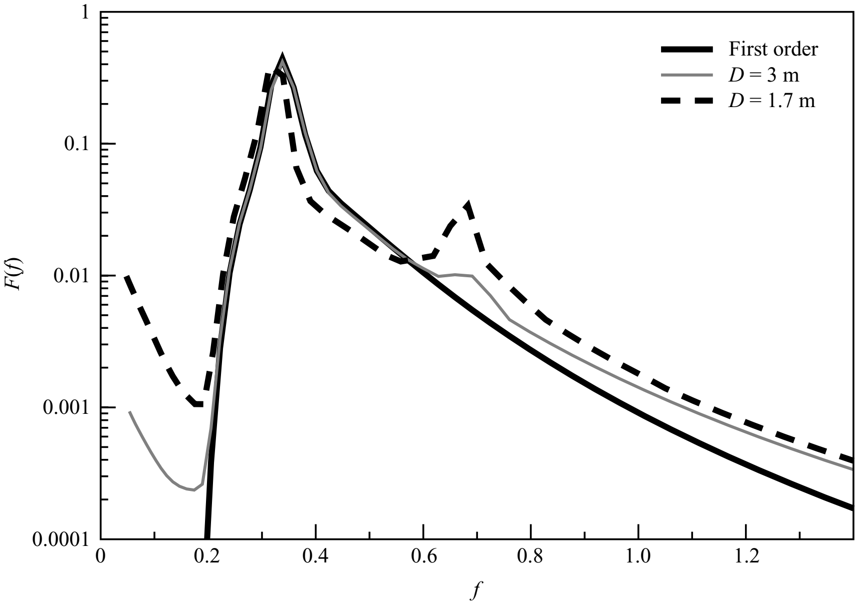

FIGURE 5. Variance spectra as function of frequency (Hz) for two different values of depth obtained from the same first-order spectrum, showing the sensitive dependence of the presence of second harmonics and the wave-induced set-down on depth.

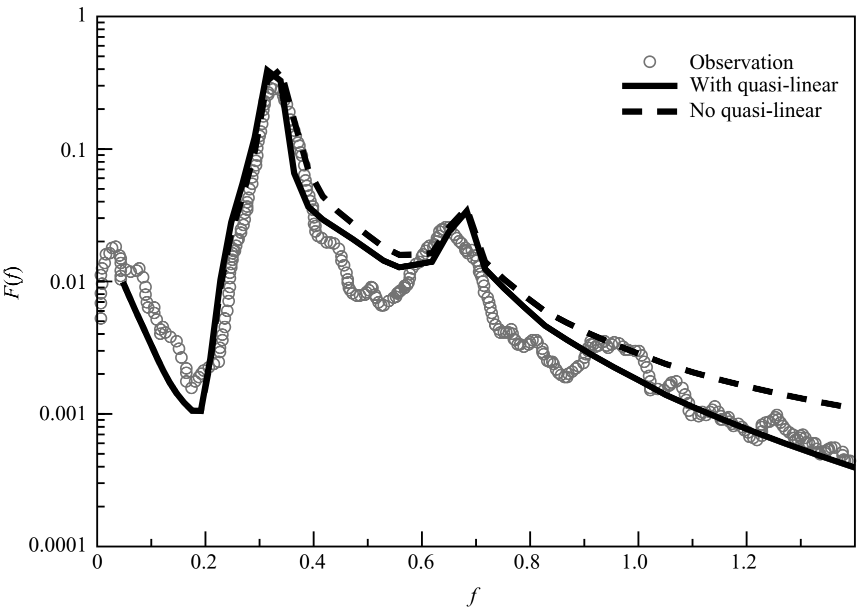

FIGURE 6. Impact of the quasi-linear term on the frequency spectrum for a depth of 1.7 m, showing a much fatter high-frequency tail when the quasi-linear term is removed. Observations obtained from Robert Jensen (personal communication, 2008) show a fairly good agreement with the second-order spectrum when the quasi-linear term is included.

digitized by Robert Jensen from the *Coastal Engineering Manual* (US Army Corps of Engineers, 2002), and they are shown in figure 6 as well. A fair agreement between the theoretical spectrum (including the quasi-linear effect) and observations is found, in particular for the high-frequency part of the spectrum. Note that the generation of second harmonics, both theoretically and experimentally, has been studied before by, for example, Norheim *et al.* (1998). These authors investigated the consequences of a stochastic formulation of the Boussinesq wave-shoaling equations, and a good agreement with observations of the wave spectrum was found. However, there was a tendency to overestimate the level of the high-frequency tail of the spectrum, and this overestimation could perhaps have been avoided by introducing the quasi-linear effect in their stochastic model. Finally, it is seen from figure 6 that the low-frequency, infra-gravity part of the spectrum is completely determined by the fully nonlinear term of (47). An extensive discussion and verification of this aspect of the second-order theory has been presented by Herbers, Elgar & Guza (1994), who have pointed out that the nonlinear term in (47) refers to the forced part of the infra-gravity waves, which is usually only a small part of the total energy in the infra-gravity range. However, using the observed bispectrum the contributions of the forced infra-gravity waves from the observed directional wave spectrum may be isolated, and a good agreement between observed forced and theoretical forced infra-gravity wave energy is obtained. For more recent work see Toffoli *et al.* (2007).

4. Skewness and kurtosis for general wave spectra

Let us now try to determine the skewness parameter C_3 and the kurtosis parameter C_4 for general wave spectra. These parameters measure deviations from the normal distribution, and this information is of relevance for certain practical applications such as the determination of the so-called sea-state bias as seen by an altimeter and the detection of extreme sea states. The skewness and the kurtosis follow from the third and fourth moments of the surface elevation p.d.f., and they are defined in this paper as follows:

$$C_3 = \frac{\mu_3}{\mu_2^{3/2}}, \ C_4 = \frac{\mu_4}{3\mu_2^2} - 1, \tag{51}$$

where $\mu_n = \langle \eta^n \rangle$, $n = 2, 3, 4$, are the second, third and fourth moments of the p.d.f. of the surface elevation, while the first moment $\langle \eta \rangle$ is assumed to vanish. For a Gaussian p.d.f. both C_3 and C_4 vanish. In order to evaluate these moments the surface elevation is expressed in terms of the Fourier integral (13), and the Fourier amplitudes are expressed in terms of the action variable A. In the next step we apply the canonical transformation (18) which is of the form $A = \epsilon a + \epsilon^2 b + \epsilon^3 c$. Hence, the moments may be expressed in terms of a, $b(a, a^*)$ and $c(a, a^*)$; hence these moments may be evaluated when the statistics for a are known. The free action variable a satisfies the Zakharov equation, and thus in principle the statistical properties of a may be obtained. We have seen that for weakly nonlinear waves it is found that in good approximation the stochastic variable a obeys Gaussian statistics, but as shown by Janssen (2003) deviations from the normal distribution are important for the dynamical evolution of the wave spectrum (due to four-wave interactions), which may result in a significant contribution to the kurtosis. However, deviations from normality are not important for the skewness of the sea surface.

The evaluation of these statistical parameters is an enormous effort, and as a first step, in § A3 the skewness and the kurtosis as obtained from the canonical transformation are determined for a single wavetrain. The single-mode result for the skewness and the kurtosis will serve as a reference for checking the general results for a spectrum of waves. These will be derived in the following sections.

4.1. *Skewness calculation*

Relatively little attention will be paid to the derivation of skewness C_3, as its general form for deep-water waves is already known (cf. Longuet-Higgins 1963; Srokosz 1986). However, the present development is given because it is a direct generalization of the deep-water result towards shallow waters.

Because of the assumption of a homogeneous sea the third moment μ_3 becomes

$$\mu_3 = \langle \eta^3 \rangle = \int \mathrm{d}\boldsymbol{k}_{1,2,3} \langle \hat{\eta}_1 \hat{\eta}_2 \hat{\eta}_3 \rangle,$$

where the Fourier transform of η is related to the action variable A through (24). Using this last equation one finds

$$\mu_3 = \int \mathrm{d}\boldsymbol{k}_{1,2,3} f_1 f_2 f_3 \left\{ \langle A_1 A_2 A_3 \rangle + 3 \langle A_1 A_2 A_3^* \rangle + \text{c.c.} \right\}.$$

In order to make progress we use the expression of the bias-corrected action variable (28), which is an expansion of the canonical transformation in terms of the small steepness ϵ. Realizing that only a result correct to fourth order in ϵ is required one finds

$$\mu_3 = \epsilon^3 \int \mathrm{d}\boldsymbol{k}_{1,2,3} f_1 f_2 f_3 \{ \langle a_1 a_2 a_3 \rangle + 3 \langle a_1 a_2 a_3^* \rangle \}$$
$$+ \epsilon^4 \int \mathrm{d}\boldsymbol{k}_{1,2,3} f_1 f_2 f_3 \{ 3 \langle a_1 a_2 \tilde{b}_3 \rangle + 6 \langle a_1 a_2^* \tilde{b}_3 \rangle + 3 \langle a_1 a_2 \tilde{b}_3^* \rangle \} + \text{c.c.}$$

Invoking now the Gaussian statistics of the free-wave action variable a it is immediately evident that the third moments such as $\langle a_1 a_2 a_3 \rangle$ vanish. In addition, using the random-phase approximation on the fourth moment (cf. (26)), the moments involving $\tilde{b}$ can all be expressed in terms of products of the action density N. Eliminating then the action density in favour of the surface-elevation spectrum E and using (32) the eventual result for the third moment becomes after setting $\epsilon = 1$

$$\mu_3 = 3 \int \mathrm{d}\boldsymbol{k}_{1,2} E_1 E_2 (\mathscr{A}_{1,2} + \mathscr{B}_{1,2}),$$

where $\mathscr{A}$ and $\mathscr{B}$ have been introduced in (34). Finally, the second moment $\mu_2 = \langle \eta^2 \rangle$ follows immediately from (50), and as only the lowest order result is required one finds

$$\mu_2 \simeq m_0 = \int \mathrm{d}\boldsymbol{k}_1 \; E_1;$$

as a consequence the skewness becomes

$$C_3 = \frac{3}{m_0^{3/2}} \int \mathrm{d}\boldsymbol{k}_{1,2} E_1 E_2 (\mathscr{A}_{1,2} + \mathscr{B}_{1,2}). \tag{52}$$

Note that this expression for the skewness holds for both deep-water and shallow-water waves. The skewness of the sea surface is, as expected, entirely determined by the sum interactions as measured by $\mathscr{A}_{1,2}$ and the difference interactions as weighted by $\mathscr{B}_{1,2}$. As a final check of the result the limit of a narrowband wavetrain in (52) was taken, i.e. $E_1 = m_0 \delta(\boldsymbol{k}_1 - \boldsymbol{k}_0)$, and it is straightforward to show that the result agrees with the expression for a single wave given in § A 3 (see (A 24)).

4.2. *Calculation of fourth moment*

Using (13) and (15) the fourth moment becomes for a homogeneous sea state

$$\mu_4 = \langle \eta^4 \rangle = \int \mathrm{d}\boldsymbol{k}_{1,2,3,4} M_{1,2,3,4} \langle A_1 A_2 A_3 A_4 + 4 A_1 A_2 A_3 A_4^* + 3 A_1 A_2 A_3^* A_4^* \rangle + \text{c.c.}, \quad (53)$$

where $M_{1,2,3,4} = (\omega_1 \omega_2 \omega_3 \omega_4)^{1/2}/4g^2$.

Now substitute the canonical transformation (28) into (53), and retain only terms up to sixth order in ϵ. The result is

$$\mu_4 = \int \mathrm{d}\boldsymbol{k}_{1,2,3,4} M_{1,2,3,4} \left\{ 3\epsilon^4 \langle a_1 a_2 a_3^* a_4^* \rangle + \epsilon^6 \left[4\langle c_1 a_2 a_3 a_4 \rangle + 12 \langle c_1 a_2 a_3 a_4^* \rangle \right.\right.$$
$$+ 4\langle c_1^* a_2 a_3 a_4 \rangle + 12\langle c_1 a_2 a_3^* a_4^* \rangle + 6 \langle a_1 a_2 \tilde{b}_3 \tilde{b}_4 \rangle + 12 \langle a_1 a_2 \tilde{b}_3 \tilde{b}_4^* \rangle + 12 \langle a_1 a_2^* \tilde{b}_3 \tilde{b}_4 \rangle$$
$$\left.\left. + 6 \langle a_1^* a_2^* \tilde{b}_3 \tilde{b}_4 \rangle + 12 \langle a_1 a_2^* \tilde{b}_3 \tilde{b}_4^* \rangle \right] + \text{c.c.} \right\}. \quad (54)$$

Clearly, there is one fourth-order term, while the remaining terms, all connected to the canonical transformation, are only sixth order in the steepness parameter ϵ. The fourth-order term has already been discussed by Janssen (2003), where it has been shown that the deviations from Gaussian statistics, as induced by the nonlinear dynamics, give rise to a kurtosis C_4 which is proportional to the square of the Benjamin–Feir index. However, all the other terms in (54) are small, and therefore only the lowest order contribution to the p.d.f., i.e. the Gaussian distribution, is required to evaluate these terms. For this reason the fourth moment consists of two parts, namely

$$\mu_4 = \mu_4^{dyn} + \mu_4^{can},$$

where a general expression for μ_4^{dyn} is given in Janssen (2003). Here we concentrate on the contribution of the canonical transformation to the fourth moment. It is fairly straightforward to evaluate the correlations involving c, using the relevant symmetries and the random phase approximation for the sixth moment, i.e.

$$\langle a_1 a_2 a_3 a_4^* a_5^* a_6^* \rangle = N_1 N_2 N_3 \left[\delta_{1-4} \left(\delta_{2-5}\delta_{3-6} + \delta_{2-6}\delta_{3-5} \right) + \delta_{1-5} \left(\delta_{2-4}\delta_{3-6} + \delta_{2-6}\delta_{3-4} \right) \right.$$
$$\left. + \delta_{1-6} \left(\delta_{2-4}\delta_{3-5} + \delta_{2-5}\delta_{3-4} \right) \right] + O(\epsilon^8).$$

Introducing one additional interaction coefficient, namely

$$\mathscr{D}_{0,1,2,3} = \frac{f_0}{f_1 f_2 f_3} \left(B^{(1)}_{0,1,2,3} + B^{(4)}_{-0,1,2,3} \right),$$

which basically represents the strength of the third harmonic and expressing the action density N in terms of the wave variance spectrum, the c terms become

$$12\epsilon^6 \int \mathrm{d}\boldsymbol{k}_{1,2,3} E_1 E_2 E_3 \left\{ \mathscr{C}_{1,1,2,2} + \frac{1}{2}\mathscr{D}_{1+2+3,1,2,3} + \frac{1}{2}\mathscr{C}_{1+2-3,1,2,3} \right\}. \quad (55)$$

The terms involving $\tilde{b}$ in (54) are a bit harder to deal with. The eventual result is

$$12\epsilon^6 \int \mathrm{d}\boldsymbol{k}_{1,2,3} E_1 E_2 E_3 \left\{ \mathscr{A}_{1,3}\mathscr{A}_{2,3} + \mathscr{B}_{1,3}\mathscr{B}_{2,3} + 2\mathscr{A}_{1,3}\mathscr{B}_{2,3} + \frac{1}{2}\mathscr{A}^2_{2,3} + \frac{1}{2}\mathscr{B}^2_{2,3} \right\}. \quad (56)$$

Combining (55) and (56) the fourth moment becomes

$$\mu_4^{can} = 3\epsilon^4 \int \mathrm{d}\boldsymbol{k}_{1,2,3} E_1 E_2 + 12\epsilon^6 \int \mathrm{d}\boldsymbol{k}_{1,2,3} E_1 E_2 E_3 \left\{ \mathscr{A}_{1,3}\mathscr{A}_{2,3} + \mathscr{B}_{1,3}\mathscr{B}_{2,3} + 2\mathscr{A}_{1,3}\mathscr{B}_{2,3} \right.$$
$$\left. + \frac{1}{2}\mathscr{A}^2_{2,3} + \frac{1}{2}\mathscr{B}^2_{2,3} + \mathscr{C}_{1,1,2,2} + \frac{1}{2}\mathscr{D}_{1+2+3,1,2,3} + \frac{1}{2}\mathscr{C}_{1+2-3,1,2,3} \right\}. \quad (57)$$

Recall that the variance is given by (50), i.e.

$$\langle \eta^2 \rangle = \int \mathrm{d}\boldsymbol{k}_1 E_1 + \int \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2 E_1 E_2 \left[\boldsymbol{\mathscr{A}_{1,2}^2 + \mathscr{B}_{1,2}^2 + 2\mathscr{C}_{1,1,2,2}}\right]. \tag{58}$$

The kurtosis parameter C_4^{can} can now be evaluated for small steepness. The result, after seting ϵ equal to one, is

$$C_4^{can} = \frac{4}{m_0^2} \int \mathrm{d}\boldsymbol{k}_{1,2,3} E_1 E_2 E_3 \left\{ (\mathscr{A}_{1,3} + \mathscr{B}_{1,3})(\mathscr{A}_{2,3} + \mathscr{B}_{2,3}) + \frac{1}{2}\mathscr{D}_{1+2+3,1,2,3} + \frac{1}{2}\mathscr{C}_{1+2-3,1,2,3} \right\}, \tag{59}$$

and this result is in agreement with the general form found by Onorato, Osborne & Serio (2008), but the coefficient inside the curly brackets was not evaluated explicitly. Here, we note that all the boldface terms in (57) and (58) cancel each other, leaving a very simple expression for C_4 indeed. Note also that all the terms in (59) have a simple physical interpretation. The nonlinear interaction coefficient $\mathscr{A}$ corresponds to the second harmonic; $\mathscr{B}$ gives the mean surface-elevation response; $\mathscr{C}$ gives the third-order correction to the amplitude of the free gravity waves; and $\mathscr{D}$ corresponds to the amplitude of the third harmonic. This interpretation becomes clearer when we take in (59) the limit of a narrowband wavetrain, i.e. $E_1 = m_0 \delta(\boldsymbol{k}_1 - \boldsymbol{k}_0)$. The result is identical to (A 23) of § A3.

Finally, the total kurtosis is given by the sum of the canonical contribution and the contribution by dynamics, i.e.

$$C_4 = C_4^{dyn} + C_4^{can}, \tag{60}$$

where C_4^{dyn} is given by (29) of Janssen (2003).

4.3. *An illustrative example*

It is of interest to evaluate the expressions for the skewness C_3 and the kurtosis C_4^{can} for a given wave spectrum and to compare the result with its narrowband limit. For the wave spectrum the very simple windsea spectrum (2) suggested by Phillips (1958) was chosen. For the Phillips spectrum the significant steepness $\epsilon = k_0 m_0^{1/2} = \alpha_p^{1/2}/2$ and the Phillips parameter $\alpha_p = 0.04$ was chosen in order to match the choice of steepness in the case of a single wavetrain discussed in § A 3. To my knowledge only for the skewness C_3 in deep water an analytical expression is known (cf. Jackson 1979 with a correction by a factor of two as pointed out by Srokosz 1986). Substituting (2) in (52) and using the one-dimensional deep-water expressions for $\mathscr{A}$ and $\mathscr{B}$ given in (43) one finds the simple expression

$$C_3 = 2\alpha_p^{1/2}, \;\; k_0 D \to \infty,$$

and for the present choice of the Phillips parameter the skewness becomes $C_3 = 0.4$. The numerical result for deep water, given in figure 7, is in perfect agreement with the analytical result. However, in general the skewness and the kurtosis can only be obtained from a numerical evaluation of (52) and (59). Figure 7 shows the skewness and the kurtosis as functions of depth for two cases. The first case has the spectrum given in (2), while the second one has a δ function spectrum of the form $E(k) = m_0 \delta(k - k_0)$ with the same variance as the first case and corresponds to the single wavetrain example of § A 3. It is clear that these two cases give a significantly different skewness and kurtosis, and hence knowledge of the spectral shape is important in determining the value of the skewness and the kurtosis.

In any event C_4^{can} is found to increase fairly rapidly as the dimensionless depth decreases when the waves approach the coast. However, the total kurtosis also has

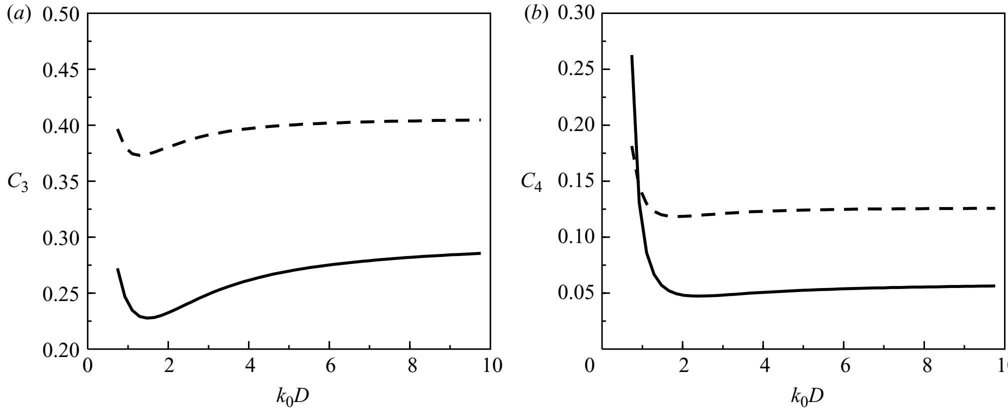

FIGURE 7. (*a*) The skewness C_3 and (*b*) the kurtosis C_4^{can} for a steepness $\epsilon = 0.1$ as functions of the dimensionless depth $x = k_0 D$. The dashed line corresponds to the case of a Phillips spectrum, and the solid line corresponds to the case of a single wavetrain with the same variance, while the carrier wavenumber equals the peak wavenumber k_0.

a contribution from the dynamics of the waves (see (60)) called C_4^{dyn}. According to Janssen & Onorato (2007), for unidirectional waves C_4^{dyn} becomes negative at around the value of the dimensionless depth $k_0 D \simeq 1.363$ which is the same point where the Stokes-frequency correction vanishes. Combining the dynamical and canonical contributions to the kurtosis it is found that the dynamical contribution dominates, and the net result is that when unidirectional waves approach the coast the kurtosis is seen to decrease with depth and even becomes negative. Hence, for unidirectional waves in shallow water the occurrence of extreme waves is less likely than in deep water. This perhaps surprising conclusion is connected to the generation of a wave-induced current and the associated mean-sea-level change in shallow water. These processes cause the vanishing of the Stokes-frequency correction at $k_0 D \simeq 1.3$ and slow down the increase of C_4^{can} with decreasing dimensionless depth (see § A 3). However, it is emphasized that for broad directional spectra the picture may be different. See the very recent numerical simulations by Toffoli *et al.* (2008, private communication) in which only for narrow directional spectra the total kurtosis is found to be negative. Broader spectra show a kurtosis which almost vanishes, corresponding to an almost Gaussian sea state.

5. Conclusions

In the Hamiltonian formulation of surface gravity waves a key role is played by the canonical transformation that eliminates the effects of non-resonant interactions on the evolution of the free-wave action variable as much as possible. Therefore, the canonical transformation provides us with an elegant method to separate the non-resonant interactions (bound waves for example) from the important resonant interactions as described by the Zakharov equation. In a wave prediction system the evolution equation for the spectrum of an ensemble of ocean waves is solved. This equation follows from the Zakharov equation and therefore gives the spectrum of the free waves. In order to obtain the actual wave spectrum one still needs to take the consequences of the canonical transformation into account.

Starting from the canonical transformation of surface gravity waves a general expression for wavenumber and directional-frequency spectrum has been obtained.

These diagnostic relations are valid for general two-dimensional spectra and may be applied both in deep and shallow waters ($kD \geqslant 1$). For the wavenumber spectrum it is found that there are two nonlinear corrections, one related to the generation of bound waves and infra-gravity waves and one quasi-linear term giving a correction to the energy of the free waves. In agreement with Creamer *et al.* (1989) when the general result is applied to the case of one-dimensional propagation, the combination of the nonlinear and quasi-linear corrections results in a small change to the first-order free-wavenumber spectrum. This contrasts with the result of Barrick & Weber (1977) for the second-order spectrum, who only considered the fully nonlinear term. This term on its own leads to divergent behaviour of the total wave spectrum. In fact, for high wavenumbers the second-order correction is more important than the first-order one, signalling that the perturbation approach would fail.

A key role in this development is played by the quasi-linear term which removes the divergent behaviour of the fully nonlinear term. In other words, a key role is played by the $B^{(2)}_{1,2,3,4}$ term of the canonical transformation. On the one hand, this terms assures that the Zakharov equation is Hamiltonian; on the other hand, this terms assures the convergent behaviour of the second-order spectrum. It is therefore important to check that the form of this term is correct. This is reported in § A 1.

The result of this work on the wavenumber spectrum is relevant for estimation of the sea-state bias as seen by an altimeter as was discussed by Elfouhaily *et al.* (1999). These authors used the second-order theory of Longuet-Higgins (1963), which is equivalent to disregarding the quasi-linear term in (36). They basically used (36) to obtain the first-order spectrum $E(\boldsymbol{k})$ from the observed wave spectrum $F(\boldsymbol{k})$. Because the quasi-linear term is disregarded, it is not a big surprise that the first-order spectrum $E(\boldsymbol{k})$ is found to deviate to a large extent from the observed spectrum. As a consequence there will be considerable deviations from the 'classical' sea-state bias results obtained by Jackson (1979) and Srokosz (1986), because these authors assumed that the first-order spectrum is approximately given by the observed spectrum. However, when retaining the quasi-linear term in (25) the differences between the first-order spectrum and the observed one are expected to be small. This work therefore justifies the approach followed by Jackson (1979) and Srokosz (1986). The directional-frequency spectrum has, compared to the wavenumber spectrum, an additional correction related to the well-known Stokes-frequency correction. In deep water the effect of the Stokes-frequency correction is usually quite small. Nevertheless, we have seen that near the peak of the spectrum this term compensates to a large extent the effect of the quasi-linear self-interaction. In shallow water, gravity waves are steeper, and as a consequence the Stokes-frequency correction has a pronounced impact on the shape of the frequency spectrum. Also, the fully nonlinear and the quasi-linear term have a considerable impact. The fully nonlinear term will give rise to forced infra-gravity waves, while the combination of the fully nonlinear term and the quasi-linear term determines second harmonics and the level of the high-frequency tail. These last two aspects of the spectral shape in shallow water have been studied extensively before (see for example Herbers *et al.* 1994; Norheim *et al.* 1998), and a good agreement with observations of the wave spectrum has been obtained, although perhaps a better agreement would have followed if the quasi-linear effect had been included.

Expressions of the skewness and kurtosis parameters, which are extensions of known results for deep-water narrowband wavetrains to the case of general spectra in waters of finite depth, were derived. These parameters are fairly sensitive to effects of the shape of the wave spectrum, and this should be relevant for statistical distributions of wave crests and the envelope of a wavetrain, for example. It is also made plausible that the kurtosis of the sea-surface elevation decreases when waves approach the coast, and this is caused by the wave-induced mean sea level which for one-dimensional wave groups is negative. Hence, for one-dimensional waves extreme sea states are less likely to occur in waters of intermediate depth ($kD \simeq 1$). Extension of this work to the case of two-dimensional propagation is desirable, as it is already known that, for example, the dynamical part of the kurtosis reduces considerably when the directional width of the wave spectrum increases (see Waseda 2006; Gramstad & Trulsen 2007). First estimates, using parameterizations of the directional effect, do suggest, however, that the conclusion that waves are less extreme in shallow waters still holds.

Discussions with J. J. Green and Miguel Onorato were very helpful in completing this laborious work. Miguel Onorato was kind enough to evaluate a number of transfer coefficients by means of Mathematica. Also, Robert Jensen was so kind as to digitize a number of shallow-water spectra from the *Coastal Engineering Manual* (US Army Corps of Engineers, 2002). Finally, Sergei Annenkov provided a semi-analytical proof that, for deep-water waves the canonical transformation conserves the wave variance.

Appendix. Remarks on Zakharov equation

A.1. *Canonical transformation*

In order to obtain the coefficients in the canonical transformation $A(a, a^*)$, given in (18), we substitute the transformation into the Hamilton equation (17), and considering weakly nonlinear waves we evaluate the resulting equation to third order in amplitude only. The time derivatives in the quadratic and cubic terms of the transformation are evaluated by means of the anticipated result (19) for the evolution in time of the free-wave canonical variable $a(\boldsymbol{k}, t)$. As only accuracy up to third order in amplitude is required we may use the linear approximation $\partial a_1/\partial t + \mathrm{i}\omega_1 a_1 = 0$.

The result is

$$\begin{aligned}\frac{\partial}{\partial t}a_1 + \mathrm{i}\omega_1 a_1 = -\mathrm{i}\int \mathrm{d}\boldsymbol{k}_{2,3}\Big\{&\left[\Delta_{1-2-3}A^{(1)}_{1,2,3} + V^{(-)}_{1,2,3}\right]a_2 a_3 \delta_{1-2-3}\\ &+\left[\Delta_{1+2-3}A^{(2)}_{1,2,3} + 2V^{(-)}_{3,2,1}\right]a_2^* a_3 \delta_{1+2-3} + \left[\Delta_{1+2+3}A^{(3)}_{1,2,3} + V^{(+)}_{1,2,3}\right]a_2^* a_3^* \delta_{1+2+3}\Big\}\\ -\mathrm{i}\int \mathrm{d}\boldsymbol{k}_{2,3,4}\Big\{&\left[Z^{(1)}_{1,2,3,4} + W^{(1)}_{1,2,3,4} + \Delta_{1-2-3-4}B^{(1)}_{1,2,3,4}\right]a_2 a_3 a_4 \delta_{1-2-3-4}\\ &+\left[Z^{(2)}_{1,2,3,4} + W^{(2)}_{1,2,3,4} + \Delta_{1+2-3-4}B^{(2)}_{1,2,3,4}\right]a_2^* a_3 a_4 \delta_{1+2-3-4}\\ &+\left[Z^{(3)}_{1,2,3,4} + 3W^{(1)}_{4,3,2,1} + \Delta_{1+2+3-4}B^{(3)}_{1,2,3,4}\right]a_2^* a_3^* a_4 \delta_{1+2+3-4}\\ &+\left[Z^{(4)}_{1,2,3,4} + W^{(4)}_{1,2,3,4} + \Delta_{1+2+3+4}B^{(4)}_{1,2,3,4}\right]a_2^* a_3^* a_4^* \delta_{1+2+3+4}\Big\},\end{aligned} \tag{A 1}$$

where $\Delta_{1-2-3} = \omega_1 - \omega_2 - \omega_3$, $\Delta_{1+2-3-4} = \omega_1 + \omega_2 - \omega_3 - \omega_4$, etc. Furthermore, the coefficients $Z^{(i)} (i = 1, 4)$ are given in terms of the second-order coefficients $V^{(\pm)}$ and $A^{(i)}$ as follows:

$$\begin{aligned}Z^{(1)}_{1,2,3,4} = 2/3\Big[&V^{(-)}_{1,2,1-2}A^{(1)}_{3+4,3,4} + V^{(-)}_{1,3,1-3}A^{(1)}_{2+4,2,4} + V^{(-)}_{1,4,1-4}A^{(1)}_{2+3,2,3}\\ &+V^{(-)}_{3,1,3-1}A^{(3)}_{-2-4,2,4} + V^{(-)}_{4,1,4-1}A^{(3)}_{-2-3,2,3} + V^{(-)}_{2,1,2-1}A^{(3)}_{-3-4,3,4}\Big],\end{aligned} \tag{A 2}$$

while

$$Z^{(2)}_{1,2,3,4} = -2\left[V^{(-)}_{1,3,1-3}A^{(1)}_{4,2,4-2} + V^{(-)}_{1,4,1-4}A^{(1)}_{3,2,3-2} + V^{(-)}_{3,1,3-1}A^{(1)}_{2,4,2-4} + V^{(-)}_{4,1,4-1}A^{(1)}_{2,3,2-3} - V^{(-)}_{1+2,1,2}A^{(1)}_{3+4,3,4} - V^{(+)}_{-1-2,1,2}A^{(3)}_{-3-4,3,4}\right] \quad (A\,3)$$

and

$$Z^{(3)}_{1,2,3,4} = 2\left[V^{(-)}_{1,4,1-4}A^{(3)}_{-2-3,2,3} - V^{(-)}_{1+2,1,2}A^{(1)}_{4,3,4-3} - V^{(-)}_{1+3,1,3}A^{(1)}_{4,2,4-2} + V^{(-)}_{4,1,4-1}A^{(1)}_{2+3,2,3} - V^{(+)}_{1,3,-1-3}A^{(1)}_{2,4,2-4} - V^{(+)}_{-1-2,1,2}A^{(1)}_{3,4,3-4}\right], \quad (A\,4)$$

and, finally,

$$Z^{(4)}_{1,2,3,4} = 2/3\left[V^{(+)}_{-1-2,1,2}A^{(1)}_{3+4,3,4} + V^{(+)}_{-1-3,1,3}A^{(1)}_{2+4,2,4} + V^{(+)}_{-1-4,1,4}A^{(1)}_{2+3,2,3} + V^{(-)}_{1+3,1,3}A^{(3)}_{-2-4,2,4} + V^{(-)}_{1+4,1,4}A^{(3)}_{-2-3,2,3} + V^{(-)}_{1+2,1,2}A^{(3)}_{-3-4,3,4}\right]. \quad (A\,5)$$

We shall comment on how $Z^{(i)}(i=1,4)$ was obtained in a short while. Let us first simplify the second-order contributions to (A 1). This is straightforward, as for gravity waves there are no resonant three-wave interactions. Then, $A^{(i)}$ can be chosen in such a way that the second-order terms vanish, and as a consequence we obtain

$$A^{(1)}_{1,2,3} = -\frac{V^{(-)}_{1,2,3}}{\omega_1-\omega_2-\omega_3}, \quad A^{(2)}_{1,2,3} = -2\frac{V^{(-)}_{3,2,1}}{\omega_1+\omega_2-\omega_3}, \quad A^{(3)}_{1,2,3} = -\frac{V^{(+)}_{1,2,3}}{\omega_1+\omega_2+\omega_3},$$

and the evolution equation for $a(\boldsymbol{k},t)$ becomes

$$\begin{aligned}\frac{\partial}{\partial t}a_1 + \mathrm{i}\omega_1 a_1 = &\\ -\mathrm{i}\int \mathrm{d}\boldsymbol{k}_{2,3,4}\Big\{&\left[Z^{(1)}_{1,2,3,4} + W^{(1)}_{1,2,3,4} + \Delta_{1-2-3-4}B^{(1)}_{1,2,3,4}\right]a_2a_3a_4\delta_{1-2-3-4}\\ &+\left[Z^{(2)}_{1,2,3,4} + W^{(2)}_{1,2,3,4} + \Delta_{1+2-3-4}B^{(2)}_{1,2,3,4}\right]a_2^*a_3a_4\delta_{1+2-3-4}\\ &+\left[Z^{(3)}_{1,2,3,4} + 3W^{(1)}_{4,3,2,1} + \Delta_{1+2+3-4}B^{(3)}_{1,2,3,4}\right]a_2^*a_3^*a_4\delta_{1+2+3-4}\\ &+\left[Z^{(4)}_{1,2,3,4} + W^{(4)}_{1,2,3,4} + \Delta_{1+2+3+4}B^{(4)}_{1,2,3,4}\right]a_2^*a_3^*a_4^*\delta_{1+2+3+4}\Big\}.\end{aligned}$$

Before we start eliminating a number of the third-order terms it is important to mention a number of ‘natural’ symmetries. These are symmetries that specify that the integrals occurring in the Hamiltonian (16) are unaffected by relabelling of the dummy integration variables. The second-order coefficient $V^{(-)}$ only satisfies symmetry with interchanging of the last indices; hence, $V^{(-)}_{1,2,3}=V^{(-)}_{1,3,2}$, while $V^{(+)}_{1,2,3}$ is symmetric under all transpositions of 1, 2 and 3. Furthermore, $W^{(1)}_{1,2,3,4}$ is therefore symmetric under the transpositions of 2, 3, 4, whereas $W^{(4)}_{1,2,3,4}$ is symmetric under transpositions of all its indices. Also, $W^{(2)}_{1,2,3,4}$ remains symmetric under transpositions within the groups (1,2) and (3,4). In addition, the coefficients should allow the Hamiltonian to be a real quantity. For the Hamiltonian (16) this gives one additional condition: $W^{(2)}_{1,2,3,4}$ should be symmetric under transpositions of the pairs (1,2) and (3,4). The coefficients occurring in the canonical transformation only enjoy a limited number of ‘natural’ symmetries; $B^{(1)}_{1,2,3,4}$ is symmetric with respect to interchanges of 2, 3 and 4, while $B^{(2)}_{1,2,3,4}=B^{(2)}_{1,2,4,3}$ and $B^{(3)}_{1,2,3,4}=B^{(3)}_{1,3,2,4}$ only. Finally, $B^{(4)}_{1,2,3,4}$ is invariant for interchanging the indices 2, 3 and 4. In the construction of $Z^{(i)}(i=1,4)$ we have made sure that they enjoy the same symmetries as $B^{(i)}(i=1,4)$.

Let us now eliminate those third-order terms that do not give rise to resonant four-wave interactions. These are the terms involving $\delta_{1-2-3-4}$, $\delta_{1+2+3-4}$ and $\delta_{1+2+3+4}$. These terms vanish when the corresponding B coefficients satisfy

$$\left.\begin{aligned} B^{(1)}_{1,2,3,4} &= -\frac{1}{\omega_1-\omega_2-\omega_3-\omega_4}\left(Z^{(1)}_{1,2,3,4}+W^{(1)}_{1,2,3,4}\right),\\ B^{(3)}_{1,2,3,4} &= -\frac{1}{\omega_1+\omega_2+\omega_3-\omega_4}\left(Z^{(3)}_{1,2,3,4}+3W^{(1)}_{4,3,2,1}\right),\\ B^{(4)}_{1,2,3,4} &= -\frac{1}{\omega_1+\omega_2+\omega_3+\omega_4}\left(Z^{(4)}_{1,2,3,4}+W^{(4)}_{1,2,3,4}\right). \end{aligned}\right\}$$

As a consequence the evolution equation for $a(\boldsymbol{k},t)$ becomes

$$\frac{\partial}{\partial t}a_1+\mathrm{i}\omega_1 a_1=-\mathrm{i}\int \mathrm{d}\boldsymbol{k}_{2,3,4}T_{1,2,3,4}a_2^*a_3a_4\delta_{1+2-3-4}, \tag{A 6}$$

where we have introduced the interaction coefficient T as

$$T_{1,2,3,4}=Z^{(2)}_{1,2,3,4}+W^{(2)}_{1,2,3,4}+\Delta_{1+2-3-4}B^{(2)}_{1,2,3,4}. \tag{A 7}$$

Finally, the determination of the term $B^{(2)}$ requires special attention because surface gravity waves enjoy resonant interaction for the combination $\Delta_{1+2-3-4}=\omega_1+\omega_2-\omega_3-\omega_4=0$. It is then not possible to simply eliminate the $\delta_{1+2-3-4}$ term. Instead, $B^{(2)}$ is determined from the requirement that also in terms of the free-wave action density we have a Hamiltonian system. Hence, we require that $T_{1,2,3,4}=T_{4,3,2,1}$ is symmetrical. Although $W^{(2)}$ is symmetric, $Z^{(2)}$ and $B^{(2)}$ are not symmetric. Therefore, T and $W^{(2)}$ may be eliminated from (A 7) by subtracting the (4,3,2,1) version of (A 7). Observing that $\Delta_{4+3-2-1}=-\Delta_{1+2-3-4}$ one finds

$$\Delta_{1+2-3-4}\left(B^{(2)}_{1,2,3,4}+B^{(2)}_{4,3,2,1}\right)=Z^{(2)}_{4,3,2,1}-Z^{(2)}_{1,2,3,4}; \tag{A 8}$$

so the asymmetry in $Z^{(2)}$ drives $B^{(2)}$. This still looks like a singular equation for $B^{(2)}$, but the remarkable thing is that for wavenumber quartets satisfying the resonance condition $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4$ the right-hand side of (A 8) (*RA8*) is proportional to $\Delta_{1+2-3-4}$. In order to see this we evaluate *RA8* by using (A 3) with the result

$$\begin{aligned} RA8 = &-2V^{(-)}_{1,3,1-3}V^{(-)}_{4,2,4-2}\left[\frac{1}{\omega_3+\omega_{1-3}-\omega_1}-\frac{1}{\omega_2+\omega_{4-2}-\omega_4}\right]\\ &-2V^{(-)}_{2,4,2-4}V^{(-)}_{3,1,3-1}\left[\frac{1}{\omega_1+\omega_{3-1}-\omega_3}-\frac{1}{\omega_4+\omega_{2-4}-\omega_2}\right]\\ &-2V^{(-)}_{1+2,1,2}V^{(-)}_{3+4,3,4}\left[\frac{1}{\omega_{1+2}-\omega_1-\omega_2}-\frac{1}{\omega_{3+4}-\omega_3-\omega_4}\right]\\ &-2V^{(+)}_{-1-2,1,2}V^{(+)}_{-3-4,3,4}\left[\frac{1}{\omega_{1+2}+\omega_1+\omega_2}-\frac{1}{\omega_{3+4}+\omega_3+\omega_4}\right]. \end{aligned}$$

Now the terms involving the angular frequencies are all proportional to $\Delta_{1+2-3-4}$. For example, the first term becomes

$$\frac{1}{\omega_3+\omega_{1-3}-\omega_1}-\frac{1}{\omega_2+\omega_{4-2}-\omega_4}=\frac{\Delta_{1+2-3-4}+\omega_{4-2}-\omega_{1-3}}{(\omega_3+\omega_{1-3}-\omega_1)(\omega_2+\omega_{4-2}-\omega_4)},$$

and for the resonance condition $\boldsymbol{k}_1+\boldsymbol{k}_2=\boldsymbol{k}_3+\boldsymbol{k}_4$ the term $\omega_{4-2}-\omega_{1-3}$ vanishes. As a consequence the singular terms $\Delta_{1+2-3-4}$ can be removed from (A 8), leaving the regular equation

$$B^{(2)}_{1,2,3,4} + B^{(2)}_{4,3,2,1} = X_{1,2,3,4} + Y_{1,2,3,4}, \tag{A 9}$$

with

$$X_{1,2,3,4} = -2A^{(1)}_{1+2,1,2}A^{(1)}_{3+4,3,4} + 2A^{(3)}_{-1-2,1,2}A^{(3)}_{-3-4,3,4}$$

and

$$Y_{1,2,3,4} = 2A^{(1)}_{2,4,2-4}A^{(1)}_{3,1,3-1} - 2A^{(1)}_{1,3,1-3}A^{(1)}_{4,2,4-2}.$$

I have grouped the terms in X and Y because of the different symmetry properties. The term X enjoys the 'natural' symmetries and the Hamiltonian property, i.e.

$$X_{1,2,3,4} = X_{4,3,2,1}, \ \ X_{1,2,3,4} = X_{1,2,4,3},$$

while Y has the Hamiltonian property but not the 'natural' symmetry property as

$$Y_{1,2,3,4} = Y_{4,3,2,1}, \ \ Y_{1,2,4,3} = -Y_{2,1,3,4},$$

but the relation $Y_{1,2,3,4} = Y_{1,2,4,3}$ does not hold. A solution of (A 9) is now constructed respecting the 'natural' symmetry $B^{(2)}_{1,2,3,4} = B^{(2)}_{1,2,4,3}$. Therefore, I have tried a solution of the type

$$B^{(2)}_{1,2,3,4} = \alpha[Y_{1,2,3,4} + Y_{1,2,4,3}] + \beta X_{1,2,4,3}, \tag{A 10}$$

and substitution of this in (A 9) gives $\alpha = 1/2$ and $\beta = 1/2$. Evidently, because (A 9) is only an equation for the symmetric part of $B^{(2)}_{1,2,3,4}$, one can always add to the solution an arbitrary asymmetric function $\lambda_{1,2,3,4}$ with the property that $\lambda_{1,2,3,4} = \lambda_{1,2,4,3} = -\lambda_{4,3,2,1}$. Although this indeterminacy will affect the solution for $a(\boldsymbol{k})$ it does not affect $A(\boldsymbol{k})$, and therefore one might as well choose $\lambda_{1,2,3,4} = 0$.

Using (A 10) and the expressions for X and Y

$$\begin{aligned} B^{(2)}_{1,2,3,4} &= 1/2[Y_{1,2,3,4} + Y_{1,2,4,3}] + 1/2X_{1,2,4,3} \\ &= A^{(1)}_{2,4,2-4}A^{(1)}_{3,1,3-1} - A^{(1)}_{4,2,4-2}A^{(1)}_{1,3,1-3} \\ &\quad + A^{(1)}_{2,3,2-3}A^{(1)}_{4,1,4-1} - A^{(1)}_{3,2,3-2}A^{(1)}_{1,4,1-4} \\ &\quad - A^{(1)}_{1+2,1,2}A^{(1)}_{3+4,3,4} + A^{(3)}_{-1-2,1,2}A^{(3)}_{-3-4,3,4}, \end{aligned} \tag{A 11}$$

while using (A 11) in the expression for $T_{1,2,3,4}$ from (A 7) one finds

$$\begin{aligned} T_{1,2,3,4} = {} & W^{(2)}_{1,2,3,4} \\ & - V^{(-)}_{1,3,1-3}V^{(-)}_{4,2,4-2}\left[\frac{1}{\omega_3 + \omega_{1-3} - \omega_1} + \frac{1}{\omega_2 + \omega_{4-2} - \omega_4}\right] \\ & - V^{(-)}_{2,3,2-3}V^{(-)}_{4,1,4-1}\left[\frac{1}{\omega_3 + \omega_{2-3} - \omega_2} + \frac{1}{\omega_1 + \omega_{4-1} - \omega_4}\right] \\ & - V^{(-)}_{1,4,1-4}V^{(-)}_{3,2,3-2}\left[\frac{1}{\omega_4 + \omega_{1-4} - \omega_1} + \frac{1}{\omega_2 + \omega_{3-2} - \omega_3}\right] \\ & - V^{(-)}_{2,4,2-4}V^{(-)}_{3,1,3-1}\left[\frac{1}{\omega_4 + \omega_{2-4} - \omega_2} + \frac{1}{\omega_1 + \omega_{3-1} - \omega_3}\right] \\ & - V^{(-)}_{1+2,1,2}V^{(-)}_{3+4,3,4}\left[\frac{1}{\omega_{1+2} - \omega_1 - \omega_2} + \frac{1}{\omega_{3+4} - \omega_3 - \omega_4}\right] \\ & - V^{(+)}_{-1-2,1,2}V^{(+)}_{-3-4,3,4}\left[\frac{1}{\omega_{1+2} + \omega_1 + \omega_2} + \frac{1}{\omega_{3+4} + \omega_3 + \omega_4}\right], \end{aligned}$$

and the energy density in terms of the 'free-wave' action variable a becomes

$$E = \int \mathrm{d}\boldsymbol{k}_1 \omega_1 a_1^* a_1 + \frac{1}{2}\int \mathrm{d}\boldsymbol{k}_{1,2,3,4} T_{1,2,3,4} a_1^* a_2^* a_3 a_4 \delta_{1+2-3-4}.$$

In summary, I have found exactly the same results as Krasitskii (1994). It should be emphasized that I have not made explicit use of the specific form of the coupling coefficients $V^{(\pm)}_{1,2,3}$ and $W^{(i)}_{1,2,3,4}$ $(i = 1, 4)$. I have only utilized their symmetry properties, and therefore, the present result is fairly general. The success of this approach depends entirely on the observation that it is possible to obtain a non-singular answer for the $B^{(2)}_{1,2,3,4}$ coefficient of the canonical transformation. In other words, there must be some deep reason why the right-hand side of (A 8) is proportional to $\Delta_{1+2-3-4}$, giving a regular equation for $B^{(2)}_{1,2,3,4}$, but I haven't been able to figure out the reason why.

Finally, an important remark regarding the canonical transformation for resonant interactions. Consider once more (A 9) which determines $B^{(2)}_{1,2,3,4}$. It is emphasized that strictly speaking we only have a condition on $B^{(2)}_{1,2,3,4}$ for non-resonant waves, namely when $\Delta_{1+2-3-4} \neq 0$. Therefore, for resonant waves the canonical transformation is arbitrary. For a continuous spectrum one may apply, however, a continuity argument to determine the canonical transformation. Clearly, (A 9) determines $B^{(2)}_{1,2,3,4}$ away from the resonance surface, but nevertheless, the relation holds arbitrarily close to the resonance. Insisting on continuity of the transformation therefore gives $B^{(2)}_{1,2,3,4}$ at the resonance surface. This has implications for the finite-amplitude expansion for a 'single' wave. Taking the narrowband limit of a continuous spectrum will therefore give a different answer than when one starts from a discrete wave from the outset.

A.2. *Nonlinear transfer coefficients*

Defining $q = \omega^2/g$ the second-order coefficients become

$$V^{(\pm)}_{1,2,3} = \frac{1}{4\sqrt{2}}\left\{[\boldsymbol{k}_1\cdot\boldsymbol{k}_2 \pm q_1 q_2]\left(\frac{g\omega_3}{\omega_1\omega_2}\right)^{1/2} + [\boldsymbol{k}_1\cdot\boldsymbol{k}_3 \pm q_1 q_3]\left(\frac{g\omega_2}{\omega_1\omega_3}\right)^{1/2} + [\boldsymbol{k}_2\cdot\boldsymbol{k}_3 + q_2 q_3]\left(\frac{g\omega_1}{\omega_2\omega_3}\right)^{1/2}\right\}$$

with $k_i = |\boldsymbol{k}_i|$, $\omega_i = \omega(k_i)$. The third-order coefficients become

$$W^{(1)}_{1,2,3,4} = \frac{1}{3}[U_{2,3,-1,4} + U_{2,4,-1,3} + U_{3,4,-1,2} - U_{-1,2,3,4} - U_{-1,3,2,4} - U_{-1,4,2,3}],$$
$$W^{(2)}_{1,2,3,4} = U_{-1,-2,3,4} + U_{3,4,-1,-2} - U_{3,-2,-1,4} - U_{-1,3,-2,4} - U_{-1,4,3,-2} - U_{4,-2,3,-1}$$
$$W^{(4)}_{1,2,3,4} = \frac{1}{3}[U_{1,2,3,4} + U_{1,3,2,4} + U_{1,4,2,3} + U_{2,3,1,4} + U_{2,4,1,3} + U_{3,4,1,2}]$$

with

$$U_{1,2,3,4} = \frac{1}{16}\left(\frac{\omega_3\omega_4}{\omega_1\omega_2}\right)^{1/2}\left[2\left(k_1^2 q_2 + k_2^2 q_1\right) - q_1 q_2 (q_{1+3} + q_{2+3} + q_{1+4} + q_{2+4})\right].$$

A.3. *Results for a single wavetrain*

Here, we study the case of a single wave, and we will derive expressions for the wave spectrum, the skewness and the kurtosis for both deep- and shallow-water waves. We also discuss the relation between the canonical transformation and the well-known Stokes expansion.

Let us apply the present formalism to the special case of a single wave. We therefore write

$$a_1 = a\delta(k_1 - k_0), \tag{A 12}$$

and the Zakharov equation (A 6) becomes

$$\frac{\partial}{\partial t}a + \mathrm{i}\omega_0 a = -\mathrm{i}T_{0,0,0,0}|a|^2 a, \tag{A 13}$$

where for arbitrary depth $T_{0,0,0,0}$ was derived by Janssen & Onorato (2007). It reads

$$T_{0,0,0,0}/k_0^3 = \frac{9T_0^4 - 10T_0^2 + 9}{8T_0^3} - \frac{1}{k_0 D}\left\{\frac{(2v_g - c_0/2)^2}{c_S^2 - v_g^2} + 1\right\},$$

where $c_0 = \omega_0/k_0$ is the phase speed, $v_g = \partial\omega/\partial k$ is the group velocity, and $c_S^2 = gD$.

The differential equation (A 13) may be solved with the ansatz $a = a_0 \exp(-\mathrm{i}\Omega_0 t)$, and as a result one finds that a_0 is a constant, while the angular frequency Ω_0 reads

$$\Omega_0 = \omega_0 + T_{0,0,0,0}|a_0|^2,$$

and the nonlinear term corresponds to the Stokes-frequency correction. The next step is to evaluate the canonical transformation $A = A(a, a^*)$. Substitution of (A 12) into (18) gives

$$\begin{aligned} A_1 = {} & A^{(2)}_{1,0,0}|a|^2\delta(k_1) + a\delta(k_1 - k_0) + A^{(1)}_{1,0,0}a^2\delta(k_1 - 2k_0) + A^{(3)}_{1,0,0}a^{*2}\delta(k_1 + 2k_0) \\ & + B^{(2)}_{1,0,0,0}|a|^2 a\delta(k_1 - k_0) + B^{(3)}_{1,0,0,0}|a|^2 a^*\delta(k_1 + k_0) \\ & + B^{(1)}_{1,0,0,0}a^3\delta(k_1 - 3k_0) + B^{(4)}_{1,0,0,0}a^{*3}\delta(k_1 + 3k_0). \end{aligned} \tag{A 14}$$

Equation (A 14) shows that apart from a mode at wavenumber k_0, A_1 has contributions at $k = \pm 2k_0$ and at $k = \pm 3k_0$ and a nonlinear correction to the linear mode at $k = \pm k_0$. In second order one also finds in general a wave-induced mean-elevation contribution (cf. Janssen & Onorato 2007) which for deep water can be shown to vanish. The surface elevation η then follows from substitution of (A 14) into

$$\eta = \int \mathrm{d}k\,\sqrt{\frac{\omega}{2g}}\,A(k)\,\mathrm{e}^{\mathrm{i}kx} + \text{c.c.},$$

and the result is, upon introduction of the surface-elevation amplitude a according to $a_0 \to (g/2\omega_0)^{1/2}a$,

$$\eta = \Delta a^2 + a(1 + \gamma a^2)\cos\theta + \alpha a^2 \cos 2\theta + \beta a^3 \cos 3\theta + \dots, \tag{A 15}$$

where α, β, γ and Δ are known functions of wavenumber and depth, and they follow from an extension of the second-order result of Janssen & Onorato (2007). Thus, the coefficients read

$$\left.\begin{aligned} \Delta &= \lim_{\epsilon\to 0}\tfrac{g}{2\omega_0} f_\epsilon\left(A^{(2)}_{\epsilon,0,0} + A^{(2)}_{-\epsilon,0,0}\right), \quad \gamma = \tfrac{g}{2\omega_0}\left[B^{(2)}_{0,0,0,0} + B^{(3)}_{-0,0,0,0}\right], \\ \alpha &= \left(\tfrac{g\omega_2}{2\omega_0^2}\right)^{1/2}\left[A^{(1)}_{2,0,0} + A^{(3)}_{-2,0,0}\right], \quad \beta = \left(\tfrac{\omega_3}{\omega_0}\right)^{1/2}\tfrac{g}{2\omega_0}\left[B^{(1)}_{3,0,0,0} + B^{(4)}_{-3,0,0,0}\right], \end{aligned}\right\}$$

where $A^{(i)}(i = 1, 3)$ and $B^{(j)}(j = 1, 4)$ are the interaction coefficients that naturally occur in the present Hamiltonian approach, and they are explicitly given in the §§ A 1 and A 2. Here we introduced a slight abuse of notation, as the index '2' now refers to wavenumber $2k_0$, etc. It is a straightforward (but laborious) task to evaluate the coupling coefficients. In deep water they become

$$B^{(1)}_{3,0,0,0} = \frac{3^{3/4}}{8}(1+\sqrt{3})\frac{k_0^3}{\omega_0},\ B^{(2)}_{0,0,0,0} = -\frac{1}{2}\frac{k_0^3}{\omega_0},\ B^{(3)}_{-0,0,0,0} = \frac{1}{4}\frac{k_0^3}{\omega_0}$$

and

$$B^{(4)}_{-3,0,0,0} = \frac{3^{3/4}}{8}(1-\sqrt{3})\frac{k_0^3}{\omega_0},$$

while

$$A^{(1)}_{2,0,0} = \frac{1}{4}\left(\frac{2g}{\omega_2}\right)^{1/2}(1+\sqrt{2})\frac{k_0^2}{\omega_0},\ A^{(3)}_{-2,0,0} = \frac{1}{4}\left(\frac{2g}{\omega_2}\right)^{1/2}(1-\sqrt{2})\frac{k_0^2}{\omega_0}.$$

Using the expression for the coupling coefficients the following canonical transformation for a single wave is found:

$$\eta/a = \left(1-\frac{\epsilon^2}{8}\right)\cos\theta + \frac{1}{2}\epsilon\cos 2\theta + \frac{3}{8}\epsilon^2\cos 3\theta, \tag{A 16}$$

where $\epsilon = k_0 a$ is the wave slope; $\theta = k_0 x - \Omega_0 t + \phi$, ϕ is the arbitrary phase of the wave; and $\Omega_0 = \omega_0(1+\epsilon^2/2)$ is the nonlinear dispersion relation.

The present weakly nonlinear expansion of the surface elevation in terms of the steepness ϵ is an example of a Stokes expansion. However, it should be noted that the Stokes expansion is not unique. This can be checked by obtaining the expansion of the surface elevation from the original Hamilton equations (17), and it can be shown that there is a whole family of solutions, parameterized by the initial condition of the first-harmonic amplitude at third order in wave steepness. Solution (A 16) belongs to this family, and clearly this is the one that is relevant in establishing a connection between the single-mode results and the narrowband limit of the result for general wave spectra. Also note that the family of Stokes solutions can be generated from the canonical transformation by using a slightly more general starting point, namely (A 12) with $a = a^{(0)} + \epsilon^2 a^{(2)}$ with $a^{(2)}$ arbitrary.

For arbitrary depth the canonical transformation for a narrowband wavetrain can be evaluated as well. After some tedious but straightforward algebra all the coupling coefficients can be eliminated in favour of wavenumber k_0 and $T_0 = \tanh x$. Hence,

$$\Delta = -\frac{k_0}{4}\frac{c_S^2}{c_S^2 - v_g^2}\left[\frac{2(1-T_0^2)}{T_0} + \frac{1}{x}\right],\ \alpha = \frac{k_0}{4T_0^3}(3-T_0^2),$$
$$\beta = \frac{3k_0^2}{64T_0^6}\left[8 + (1-T_0^2)^3\right],\ \gamma = -\frac{1}{2}\alpha^2, \tag{A 17}$$

where $x = k_0 D$; $T_0 = \tanh x$; $c_S^2 = gD$; $v_g = \partial\omega/\partial k$; $\omega = (gk_0T_0)^{1/2}$. These results were checked against calculations of the coupling coefficients on the computer. Furthermore, the deep-water limit is in agreement with the known results given in (A 16).

In order to derive expressions for the wave spectrum, the wave variance, the skewness and the kurtosis of a random, narrowband wavetrain we have to make the assumption that the sea state is Gaussian and homogeneous. For a narrowband wavetrain, normality of the p.d.f. of the linear wave implies that the phase is uniformly distributed, while the amplitude a obeys the Rayleigh distribution. Here, a will be scaled with $\sigma = \sqrt{m_0}$ so that the p.d.f. of a becomes simply

$$p(a) = a\, \mathrm{e}^{-\frac{1}{2}a^2},$$

while the phase is uniformly distributed; hence

$$p(a, \theta) = \tfrac{1}{2\pi}\, a\, \mathrm{e}^{-\frac{1}{2}a^2}.$$

Because of the presence of the wave-induced mean level, the average of η is not zero. In agreement with experimental practice, we subtract the mean level $\langle\eta\rangle$. In addition, in (A 15) we scale amplitude a with σ, and we treat σ as a small parameter. Hence the surface elevation becomes

$$\eta = \Delta\sigma^2(a^2 - \langle a^2\rangle) + \sigma a(1 + \gamma\sigma^2 a^2)\cos\theta + \alpha\sigma^2 a^2 \cos 2\theta + \beta\sigma^3 a^3 \cos 3\theta + \cdots, \quad \text{(A 18)}$$

and now $\langle\eta\rangle$ vanishes. Nevertheless, nonlinear quantities such as the second moment $\langle\eta^2\rangle$ will depend on the parameter Δ (which measures the strength of the wave-induced mean sea level), as for $m > 1$, $\langle(a^2 - \langle a^2\rangle)^m\rangle$ does not vanish. Let us first evaluate the wave spectrum for a homogeneous sea, which is essentially a quadratic quantity. To that end we evaluate the spatial correlation function $\langle\eta(x + r)\eta(x)\rangle$ assuming homogeneity. The spectrum $F(k)$ then follows by taking the Fourier transform with respect to distance r. Now, since $\langle a^2\rangle = 2$, $\langle a^4\rangle = 8$ and $\langle a^6\rangle = 48$, the spectrum becomes up to fourth order in σ

$$F(k) = \tfrac{1}{2}\sigma^2(1 + 8\sigma^2\gamma)\delta(k - k_0) + 2\sigma^4[\Delta^2\delta(k) + \alpha^2\delta(k - 2k_0)] + k \rightarrow -k, \quad \text{(A 19)}$$

and it can be verified that in the deep-water limit this result agrees with the narrowband limit of the spectral approach (cf. (36)). In the general case we see that the canonical transformation will give rise to a second-harmonic peak, a correction to the energy of the first harmonic and also a contribution to zero mean wavenumber. It is left as an exercise for the reader that for finite depth the general result (A 19) also agrees with the narrowband result obtained from (36). Just like in the main text, the determination of the frequency spectrum requires special attention. In particular the Stokes-frequency correction will affect the spectral shape, and for a discussion on this see Janssen & Komen (1982).

The skewness C_3 and the kurtosis C_4 are defined as

$$C_3 = \langle\eta^3\rangle/\langle\eta^2\rangle^{3/2},\ C_4 = \langle\eta^4\rangle/3\langle\eta\rangle^2 - 1; \quad \text{(A 20)}$$

hence we need to evaluate the third and fourth moments of the p.d.f.,

$$\langle\eta^3\rangle = \int \eta^3\ p(a, \theta)\mathrm{d}a\, \mathrm{d}\theta,\ \ \langle\eta^4\rangle = \int \eta^4\ p(a, \theta)\mathrm{d}a\, \mathrm{d}\theta,$$

up to the required order in σ^2, while we also need the second moment. The last follows immediately from an integration of the wavenumber spectrum, and as a result one finds

$$\langle\eta^2\rangle = \sigma^2 + 4\sigma^4(2\gamma + \alpha^2 + \Delta^2). \quad \text{(A 21)}$$

In order to determine the skewness parameter we need to evaluate the third moment up the order σ^4. Using the expression for the surface elevation (A 18) one finds

$$\eta^3 = \sigma^3\left\{a^3\cos^3\theta + 3\sigma a^2\left[\alpha a^2 \cos 2\theta \cos^2\theta + \Delta(a^2 - \langle a^2\rangle)\cos^2\theta\right]\right\} + O(\sigma^5).$$

We perform the averaging over the angle θ first. With $\langle\cos^3\theta\rangle = 0$, $\langle\cos^2\theta\rangle = 1/2$ and $\langle\cos 2\theta\cos^2\theta\rangle = 1/4$ one finds

$$\langle\eta^3\rangle = 3\sigma^4\left[\tfrac{1}{4}\alpha\langle a^4\rangle + \tfrac{1}{2}\Delta\left(\langle a^4\rangle - \langle a^2\rangle^2\right)\right].$$

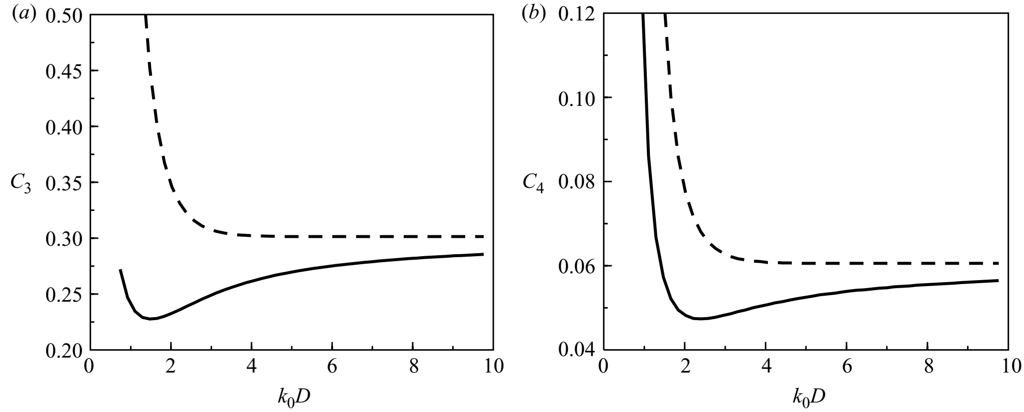

FIGURE 8. (*a*) The skewness C_3 and (*b*) the kurtosis C_4 related to the canonical transformation for a steepness $\epsilon = 0.1$ as functions of the dimensionless depth $x = k_0 D$. The black line denotes results including the wave-induced set-down, and the dashed black line denotes results without the wave-induced set-down.

Now, since $\langle a^2 \rangle = 2$ and $\langle a^4 \rangle = 8$ the third moment becomes

$$\langle \eta^3 \rangle = 6\sigma^4 (\alpha + \Delta),$$

and to the lowest significant order the skewness becomes

$$C_3 = 6\sigma (\alpha + \Delta). \tag{A 22}$$

In a similar vein the kurtosis parameter can be obtained. In order to get non-trivial results an evaluation of the fourth moment up to σ^6 is required. Now,

$$\eta^4 = \sigma^4 a^4 (1 + 4\gamma\sigma^2 a^2)\cos^4\theta + 4\sigma^5 a^3 \cos^3\theta[\Delta(a^2 - \langle a^2\rangle) + \alpha a^2 \cos 2\theta + \sigma\beta a^3 \cos 3\theta] \\ + 6\sigma^6 a^2 \cos^2\theta[\Delta^2(a^2 - \langle a^2\rangle)^2 + 2\alpha\Delta a^2(a^2 - \langle a^2\rangle)\cos 2\theta + \alpha^2 a^4 \cos^2 2\theta] + O(\sigma^7).$$

Perform the averaging over θ first. To that end we need to know some additional integrals:

$\langle \cos^4\theta \rangle = \frac{3}{8}$, $\langle \cos^3\theta \rangle = 0$, $\langle \cos^3\theta \cos 2\theta \rangle = 0$, $\langle \cos^3\theta \cos 3\theta \rangle = \frac{1}{8}$, $\langle \cos^2\theta \cos^2 2\theta \rangle = \frac{1}{4}$.

This gives

$$\langle \eta^4 \rangle = \frac{3}{8}\sigma^4 \langle a^4 \rangle + \sigma^6 \Bigg[\langle a^6 \rangle \left\{ \frac{3}{2}\left(\frac{\beta}{3} + \gamma + \alpha^2\right) + 3(\Delta^2 + \alpha\Delta) \right\} \\ - \langle a^4 \rangle \langle a^2 \rangle (6\Delta^2 + 3\alpha\Delta) + 3\Delta^2 \langle a^2 \rangle^3 \Bigg].$$

Now, since $\langle a^2 \rangle = 2$, $\langle a^4 \rangle = 8$ and $\langle a^6 \rangle = 48$, one finds

$$\langle \eta^4 \rangle = 3\sigma^4 + 24\sigma^6[\beta + 3(\gamma + \alpha^2) + 3\Delta^2 + 4\alpha\Delta].$$

Finally, by means of the expression for the variance (A 21) the kurtosis becomes to the lowest significant order

$$C_4 = 8\sigma^2[\beta + \gamma + 2(\alpha + \Delta)^2]. \tag{A 23}$$

Hence, referring to (A 17) we have now explicit expressions for the skewness and the kurtosis of a narrowband wavetrain in terms of the wave variance, wavenumber and depth. In particular, for deep water one finds (see e.g. Mori & Janssen 2006)

$$C_3 = 3\epsilon, \; C_4 = 6\epsilon^2, \tag{A 24}$$

where $\epsilon = k_0\sigma$ is the 'significant' steepness.

Finally, it is of interest to study the importance of the wave-induced mean level to the statistical properties of the sea surface. As for a wave group one typically has a set-down, and as for the range of the dimensionless depth $x \simeq 1$ $|\Delta| < \alpha$ it is seen from (A 22) and (A 23) that a set-down will give rise to a reduction of the skewness and the kurtosis. This is illustrated in figure 8 for both the skewness and the kurtosis plotted as functions of the dimensionless depth k_0D. First of all we see that there is a dramatic increase of these higher-order statistics when moving into shallower water, but this increase is significantly slowed down when effects of the wave-induced set-down are included.

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